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AMERICAN SOCIETY OF CIVIL ENGINEERS

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PAPERS

BACK-WATER AND DROP-DOWN CURVES FOR UNIFORM CHANNELS

By Nagaho Mononobe [1], M. Am. Soc. C. E.

Synopsis

Most of the existing formulas for determining back-water and drop-down curves in open channels, are applicable only to channels of simple geometric cross-section. Moreover, these formulas usually yield inaccurate results because in their derivation the effects of variation of velocity head are neglected, or the coefficient, *C*, in Chezy's mean-velocity formula is considered constant. In the present study, rational back-water and drop-down formulas are derived in such a manner that they are applicable to a wide variety of commonly used cross-sections. These formulas are based on accurate mean-velocity expressions of the exponential type, and make full allowance for velocity-head effects. The results of numerous tests in experimental channels are presented, and these results are compared with values computed by existing formulas and by the new formulas derived by the writer. In a thorough treatise entitled "Hydraulics of Open Channels", B. A. Bakhmeteff, M. Am. Soc. C. E., has also introduced the hydraulic exponent to express the characteristics of a channel cross-section. The writer's manuscript was prepared independently and has the merit of simplifying the computation in its application to various cases. This statement does not imply a criticism of Professor Bakhmeteff's treatment, but is intended merely to establish a line of demarcation, in the hope that discussers will avoid confusing the two treatments.

1.—Fundamental Formulas for Back-Water and Drop-Down Curves

In a channel having constant discharge, uniform cross-section, and uniform slope, the flow is said to be non-uniform if the water surface is not parallel to the channel bed. In this case the longitudinal profile of the water surface takes the form of a curve. If the flow is of the "tranquil" type this curve theoretically extends up stream an infinite distance and gradually approaches

Note.—Discussion on this paper will be closed in **September, 1936**, *Proceedings*.

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the straight-line profile for uniform flow at the given discharge If concave upward the curve is called a back-water curve, and if concave downward it is called a drop-down curve. However, the same differential equation is applicable to both cases, the difference being only in the boundary conditions. The determination of the surface profile for cases of non-uniform flow is a problem of much practical importance.

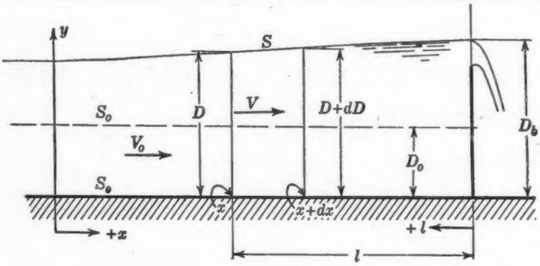


FIG. 1.

Placing the origin of the co-ordinates at any section of the channel, with the X-axis along the lowest line of the channel bed (positive in the downstream direction), and the Y-axis perpendicular to the X-axis, the general equation of motion for non-uniform flow is expressed as follows (see Fig. 1):

$$S = S_0 - \frac{dD}{dx} = \frac{V^2}{C^2 R} + \alpha \frac{d}{dx}\left(\frac{V^2}{2g}\right) \quad \text{(1)}$$

in which S = slope of the water surface when the flow at Section x is non-uniform; S_0 = slope of the channel bed = slope of the water surface for uniform flow; D = depth of water at Section x for non-uniform flow; D_0 = depth of water for uniform flow; D_b = depth of water at weir; D_s = depth of water at point of maximum drop-down; R = hydraulic radius at Section x; C = a coefficient in the velocity formula; α = a correction factor for velocity head $= 1 + \alpha' =$, say, $\frac{10}{9}$.

A general type of mean velocity formula is as follows:

$$V = C\ R^m\ S^p{}_f = C\ R^m\ S^{0.5}{}_f \quad \text{(2)}$$

in which $p = 0.5$ in most cases, and S_f = the slope, S, defined by the frictional head for V and R.

In the Chezy velocity formula, C varies considerably with the depth of water and, if Equation (2) is substituted in Equation (1),

$$S = S_0 - \frac{dD}{dx} = \frac{V^2}{C^2\ R^{2m}} + \alpha\ \frac{d}{dx}\left(\frac{V^2}{2g}\right) \quad \text{(3)}$$

or,

$$S = \frac{Q^2}{C^2\ R^{2m}\ A^2} + \alpha\ \frac{Q^2}{2g}\ \frac{d}{dx}\left(\frac{1}{A^2}\right) \quad \text{(4)}$$

in which Q = discharge; and A = hydraulic area.

A relation between D and x can be found by solving either Equation (3) or Equation (4), if S_0 and C do not vary. Since they do vary for rivers and

streams, Equations (3) and (4) are not strictly applicable to natural water-courses. However, if the stream is considered as divided into several reaches, in each of which S_0 and C are nearly uniform, and if the relation between D and x is applied to each reach separately, back-water curves can be obtained more conveniently from either Equation (3) or Equation (4), than from Equation (1), since the latter must be solved by trial.

2.—Various Formulas for Back-Water Curves

The discharge in the channel, Q, being constant, S in the velocity formula (Equation (2)) is expressed in terms of the hydraulic radius, R, corresponding to the depth of water, D, and of the roughness coefficient or velocity coefficient; $\frac{d}{dx}\left(\frac{V^2}{2\,g}\right)$ can be expressed in terms of the hydraulic area, A, and for a fixed section, A and R can be expressed in terms of D, the depth of water. Consequently, Equation (4) is a differential equation that combines D with x, its type being dependent on the relation expressed in Equation (2), and on the shape of the cross-section. Hence, the back-water curve derived by integration is necessarily subject to these two factors.

In determining the back-water and drop-down curves, therefore, it is essential to adopt the velocity formula most appropriate for the channel under consideration and to establish the relation between D and R as accurately as possible. Many of the existing formulas are deficient in one or both of these requirements and, moreover, these formulas do not include the term which expresses a change of velocity head. When the variation of mean velocity is wide, the latter omission introduces a serious error, especially in the case of drop-down curves.

Table 1 contains various types of velocity formulas and channel sections adapted to the determination of back-water and drop-down curves in current use.

TABLE 1.—Types of Existing Back-Water Formulas

Item No. (1)	Investigation (2)	Year of publication (3)	Velocity formula, V, equals: (4)	Type of channel section (5)	Variations in velocity head: (6)	Ratios of $\frac{d}{d_0}$: Back-water curve (7)	Drop-down curve (8)
1	Grashof-Bresse.	1860	$C\sqrt{D_0\,S_0}$	Broad rectangle	Considered	$\frac{1}{0.999}$ to $\frac{1}{0.07}$	0.999 to 0.00
2	Dupuit-Masoni	1863	$C\sqrt{R\,S_0}$	Common rectangle	Considered	None	None
3	Rühlmann.....	1880	$C\sqrt{D_0\,S_0}$	Broad rectangle...	Neglected	1.01 to 6.0	0.99 to 0.00
4	Tolkmitt......	1881	$C\sqrt{R\,S_0}$	Broad parabola...	Considered	1.005 to 5.0	0.995 to 0.00
5	Schaffernak....	1913	$C D_0^{0.75}\,S_0^{0.5}$	Broad rectangle...	Neglected	1.013 to 5.0	0.2 to $\frac{1}{1.04}$
6	Ehrenberger...	1914	$\begin{cases}23.78\;D_0^{0.776}\;S^{0.458}\\22.11\;D_0^{0.58}\;S_0^{0.42}\end{cases}$	Broad rectangle...	Neglected	1.000 to 5.0	None
7	Baticle........	1921	$\left\{Q^2 \text{ varies as } S\,z^5 = S_0\,z_0^5.\right.$	Approximate trapezoid	Neglected	1.001 to 10.0	None
8	Kozeny.......	1928	$C D_0^{0.7}\;S_0^{0.5}$	Broad rectangle...	Considered	None	0.10 to 0.95
9	Schoklitsch....	1930	$C D^{m}_0\;S^{0.5}_0$	Broad rectangle...	Neglected	1.01 to 5.0	None

Most of the expressions for mean velocity in Column (4), Table 1, are considerably in error because the characteristics of each section of the actual channel vary. Furthermore, in some cases (Items Nos. 3, 5, 6, 7, and 9, Table 1), the important variations in velocity head are neglected. In Item No. 7, Table 1, the symbols, z and z_0, denote $A^2 R$ and z in uniform flow, respectively.

3.—General Solutions

Existing methods of calculating back-water and drop-down curves are scarcely accurate enough for practical purposes; furthermore, the hydraulic theory upon which they are based is very imperfect. The reasons why existing methods are unsatisfactory are, as follows:

Case (A).—In order to be able to express the back-water and drop-down curves by functions having a finite number of terms, some of these methods use the Chezy velocity formula with constant C, and consider only broad, shallow, channel sections. Generally, it is impossible to express the back-water and drop-down curves by a function having a finite number of terms unless $\frac{1}{m}$ in Equation (2) is an integer and the broad section is a rectangle.

Case (B).—When m has any arbitrary value and when the channel has a broad shallow section, the back-water and drop-down curves may be expressed by functions involving infinite series; but the calculation of the numerical table (Columns (7) and (8), Table 1) is quite complicated in comparison with Case (A). The Schaffernak (Item No. 5)[2] and Schoklitsch (Item No. 9)[3] formulas are the simplest in this case. In the Rühlmann formula (Item No. 3)[4] an infinite series is used, laboriously and needlessly, in spite of the fact that the formula should belong to Case (A). In the cases of most channels with cross-sections that are not broad in comparison with their depths, the determination of the back-water and drop-down profiles by rigorous mathematical methods involves difficulties which are practically insurmountable.

Unless the Chezy formula is used, the back-water function generally requires the use of infinite series for its expression. Moreover, if the change in the velocity head is considered, two infinite series are derived from one kind of channel. In this case it is extremely laborious to construct tables of the function for all types of sections.

In the following, the writer derives some exponential formulas for the hydraulic area, wetted perimeter, and hydraulic radius, which are sufficiently accurate for practical purposes. The change in velocity head is taken into consideration, and the most appropriate velocity formula is used for each channel. In every case the desired back-water or drop-down function is expressed in terms of two sets of series.

Case 1.—The Most General Case of Non-Uniform Flow.—The hydraulic area, A, the wetted perimeter, P, the hydraulic radius, R, and the average

[2] "Hydraulik", by Forchheimer, Third Edition, p. 208.
[3] "Wasserbau", by Schoklitsch, 1930, Vol. 1, p. 103.
[4] "Hydraulik", by Forchheimer, Third Edition, p. 207.

velocity, V, are expressed most generally as functions of D, the depth of water measured from the lowest line of the channel bed; thus, with Equation (2):

$$A = a_0 + aD^s \text{..........................} (5a)$$

$$P = b_0 + bD^k \text{..........................} (5b)$$

and,

$$R = \frac{A}{P} \text{..........................} (5c)$$

in which the subscript, 0, refers the symbol to the condition of uniform flow.

Since the amount of flow in the channel is unchanged, $Q = VA = V_0 A_0$ and, therefore:

$$S_f = \frac{1}{C^2}\left(\frac{V_0 A_0}{A}\right)^2 R^{-2m} \text{....................} (6)$$

in which $\dfrac{V_0^2}{2g}\left(\dfrac{A_0}{A}\right)^2 = \dfrac{V^2}{2g}$. By substituting Equations (2) and (5) into the general differential Equation (1):

$$S_0 - \frac{dD}{dx} = \frac{V_0^2}{C^2}\left(\frac{A_0}{A}\right)^2 R^{-2m} + \alpha \frac{V_0^2}{2g}\frac{d}{dx}\left(\frac{A_0}{A}\right)^2 \text{.........} (7)$$

Replacing A, R, and P, by their values in Equations (5) and writing $\dfrac{a}{a_0} = a_1$; and $\dfrac{b}{b_0} = c$:

$$S_0 - \frac{dD}{dx} = S_0\left(\frac{1 + a_1 D_0^s}{1 + cD_0^k}\right)^{2m+2}\left(\frac{1 + cD^k}{1 + cD_0^k}\right)^{2m}$$

$$- \frac{\alpha C^2 S_0}{g}\left(\frac{1 + a_1 D_0^s}{1 + cD_0^k}\right)^{2m} S_{a1}\left(\frac{1 + a_1 D_0^s}{1 + a_1 D^s}\right)^2 \frac{D^{s-1}}{1 + a_1 D^s}\frac{dD}{dx} \text{......} (8)$$

Equation (8) is the most general differential equation that expresses the back-water and drop-down functions; but integration of the equation and the calculation of numerical tables become extremely complicated with the expression in this form, since each term in the right-hand side of the equation expands into a double series with respect to D.

Case (2).—In Which the Hydraulic Area, A, Is Monomial.—In common channels or rivers, if D is measured from the lowest line, the monomial expression, $A = aD^s$, for hydraulic area is sufficiently accurate even when $a_0 = 0$. Therefore, when $A = aD^s$; $P = b_0 + bD^k$; and,

$$R = \frac{A}{P} = \frac{a\,D^s}{b_0(1 + cD^k)} \text{......................} (9)$$

then,

$$\left(\frac{A_0}{A}\right)^2 = \left(\frac{D_0}{D}\right)^{2s} \text{......................} (10a)$$

$$S_f = \frac{V_0^2}{C^2}\left(\frac{A_0}{A}\right)^2 R^{-2m} = S_0\left(\frac{D_0}{D}\right)^{2s(m+1)}\left(\frac{1 + c\,D^k}{1 + cD^k_0}\right)^{2m} \ldots\ldots (10b)$$

and,

$$\frac{V_0^2}{2g}\,\frac{d}{dx}\left(\frac{A_0}{A}\right)^2 = -\,\frac{S_0\,C^2\,S}{g}\left(\frac{c_0 + cc_0D^k_0}{D^s_0}\right)^{-2m}\left(\frac{D_0}{D}\right)^{2s}\frac{1}{D}\,\frac{dD}{dx}\ .(10c)$$

in which $c = \dfrac{b}{b_0}$; and, $c_0 = \dfrac{b_0}{a}$.

Let,

$$\frac{S_0\,a\,C^2\,S}{g}\left(\frac{c_0 + cc_0\,D^k}{D^s_0}\right)^{-2m} = D_k \ldots\ldots\ldots\ldots\ldots (11)$$

Then, Equation (7) becomes,

$$S_0 - \frac{dD}{dx} = S_0\left(\frac{D_0}{D}\right)^{2s(m+1)}\left(\frac{1 + cD^k}{1 + cD^k_0}\right)^{2m} - \frac{D_k}{D_0}\left(\frac{D_0}{D}\right)^{2s+1}\frac{dD}{dx}$$

and, therefore,

$$S_0\,dx = \frac{\left[1 - \dfrac{D_k}{D_0}\left(\dfrac{D_0}{D}\right)^{2s+1}\right]dD}{1 - \left(\dfrac{D_0}{D}\right)^{2s(m+1)}\left(\dfrac{1 + cD^k}{1 + cD^k_0}\right)^{2m}} = F\,(D)\,dD \ldots\ldots (12)$$

The second term in the denominator of Equation (12) may be expressed by a convergent infinite series, and, as a consequence, the function, $F(D)$, can be expressed in terms of a double series of $\dfrac{D}{D_0}$; but the labor for calculating the numerical table is far greater than in Case (3), which follows.

Case (3).—In Which the Hydraulic Area, A, and Wetted Perimeter, P, Are Monomials.—In this case the differential equations become reasonably simple, since,

$$A = aD^s \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13a)$$

$$P = bD^k \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13b)$$

and,

$$R = \frac{A}{P} = \frac{a}{b}\,D^{s-k} \ldots\ldots\ldots\ldots\ldots\ldots (13c)$$

It seems somewhat irrational to express P as a monomial function of D. This is especially true in the case of sections such as the rectangle or trapezoid, in which the base width has a finite value. However, in the cases of these sections, as in the cases of most other sections in common use, actual trial shows that it is possible to choose values of b and k in Equation (13b) that will make this equation express the value of P with reasonable accuracy over a considerable range of depths. The fact that the use of this monomial expression for P does not introduce errors of appreciable magnitude into the computation of back-water and drop-down profiles, will be made evident by reference to the many experiments described subsequently herein.

4.—General Formulas for Back-Water and Drop-Down Curves

Referring to Case (3), Article 3, and following the form of Equation (13):

$$\frac{A}{A_o} = \left(\frac{D}{D_o}\right)^s; \frac{P}{P_o} = \left(\frac{D}{D_o}\right)^k; \frac{R}{R_o} = \left(\frac{D}{D_o}\right)^{s-k};$$ and $V_o = C\,R^m{}_o\,S^{0.5}{}_o$. Consequently,

$$\frac{V^2_o}{C^2}\left(\frac{A_o}{A}\right)^2 R^{-2m} = S_o\left(\frac{D}{D_o}\right)^{-2s}\left(\frac{R_o}{R}\right)^{2m} = S_o\left(\frac{D}{D_o}\right)^{-2s-2m(s-k)} \quad \ldots\ldots(14)$$

and,

$$\alpha\,\frac{V^2_o}{2g}\,\frac{d}{dx}\left(\frac{A_o}{A}\right)^2 = -\,\alpha\,\frac{V^2_o}{g} \times s\left(\frac{D}{D_o}\right)^{-2s-1}\frac{d}{dx}\left(\frac{D}{D_o}\right)$$

$$= -\,K\,D_o\left(\frac{D}{D_o}\right)^{-2s-1}\frac{d}{dx}\left(\frac{D}{D_o}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots(15)$$

in which $K = \alpha\, s\,\frac{V^2_o}{g\,D_o}$.

Back-Water.—Substituting Equations (14) and (15) into the general differential Equation (1):

$$S_o - \frac{dD}{dx} = S_o\left(\frac{D}{D_o}\right)^{-2s-2m(s-k)} - K D_o\left(\frac{D}{D_o}\right)^{-2s-1}\frac{d}{dx}\left(\frac{D}{D_o}\right)$$

Hence, by making $\frac{D}{D_o}$ equal to y, the quantity, $d\left(\frac{D}{D_o}\right)$, will equal $dy = \frac{dD}{D_o}$; and, finally,

$$\frac{S_o}{D_o}\,dx = \frac{dy}{1 - y^{-2s-2m(s-k)}} - K\,\frac{y^{-(2s+1)}}{1 - y^{-2s-2m(s-k)}}\,dy \quad \ldots\ldots\ldots(16)$$

Since m is dependent upon the velocity formula used, and s and k are determined by the channel section:

$$\frac{S_o\,dx}{D_o} = \frac{dy}{1 - y^{-r}} - K\,\frac{y^{-(2s+1)}}{1 - y^{-r}}\,dy = \frac{y^r}{y^r - 1}\,dy - K\,\frac{y^{(r-2s)-1}}{y^r - 1}\,dy \ldots(17)$$

in which $r = 2s + 2m\,(s - k)$. Integrating Equation (17) from x to $x + l$ (that is, from D to D_b), the distance, l, from the weir to the point where the depth of water, D, is known, and the rise of the water surface is $D - D_o$ (see Fig. 2):

$$\frac{S_o\,l}{D_o} = \Phi_1\left(\frac{D_b}{D_o}\right) - \Phi_1\left(\frac{D}{D_o}\right) - K\left[\Phi_2\left(\frac{D_b}{D_o}\right) - \Phi_2\left(\frac{D}{D_o}\right)\right] \ldots(18)$$

in which $D_b = D_{x+l}$; $D = D_x$; $K = \alpha s\,\frac{V^2_o}{g\,D_o}$; $\alpha = \frac{10}{9}$; $\Phi_1 = \int \frac{y^r}{y^r - 1}\,dy$; $\Phi_2 = \frac{y^{(r-2s)-1}}{y^r - 1}\,dy$; and $y = \frac{D}{D_o} > 1$.

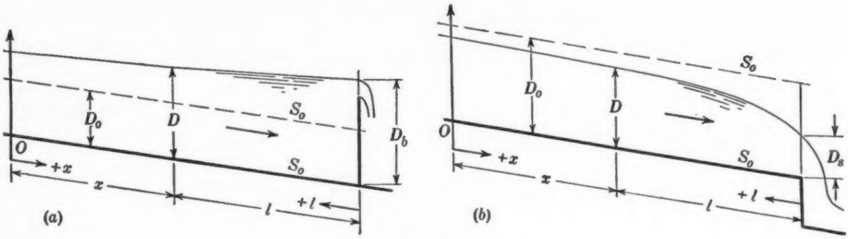


FIG. 2.

Theoretically, the influence of back-water extends backward from the weir an infinite distance. At the point where $l = \infty$, $D = D_0$, or $y = l$. Since both Φ_1 and Φ_2 become infinitely great at this point, their limits of integration are taken to be $y = y_1 = 1.001$; that is, $\frac{D}{D_0} = 1.001$

Drop-Down.—In this case, $\frac{D}{D_0} < 1$, and for convenience in integration let $z = \frac{1}{y} = \frac{D_0}{D} > 1$. Then, Equation (16) becomes:

$$\frac{S_0\,dx}{D_0} = -\frac{z^{-2}\,dz}{1 - z^r} + K\frac{z^{2s-1}\,dz}{1 - z^r} = \frac{z^{-2}\,dz}{z^r - 1} - K\frac{z^{2s-1}\,dz}{z^r - 1} \quad \ldots\ldots.(19)$$

In Equation (19), $r > 1$; and $z > 1$, and both terms on the right-hand side can be developed into a convergent series of z. Integrating from x to $x + l$ (that is, from D to D_s):

$$\frac{S_0\,l}{D_0} = \Psi_1\left(\frac{D_0}{D_s}\right) - \Psi_1\left(\frac{D_0}{D}\right) - K\left[\Psi_2\left(\frac{D_0}{D_s}\right) - \Psi_2\left(\frac{D_0}{D}\right)\right] \quad \ldots.(20)$$

in which $\Psi_1(z) = \Psi_1\left(\frac{D_0}{D}\right) = \int \frac{z^{-2}}{z^r - 1}\,dz$; and, $\Psi_2(z) = \Psi_2\left(\frac{D_0}{D}\right) \int \frac{z^{2s-1}}{z^r - 1}\,dz$.

In this case, also, Ψ_1 and Ψ_2 become infinitely great when $z = 1$, and the lower limit of the integration is taken as $z = \frac{D_0}{D} = 1.001$, or $y = \frac{D}{D_0} = 0.999$.

5.—CALCULATION OF BACK-WATER FUNCTIONS

The Back-Water Curve.—In the expression, $r = 2s + 2m\,(s - k)$, $s =$ the numerical value of the power in the function for hydraulic area; $m =$ the exponent of hydraulic radius in the velocity formula; and $k =$ the exponent of the wetted perimeter in the velocity formula. For open channels and conduits, $1 < r < 6$ and $0.5 < s \leqq 2.0$. The quantity, $\frac{1}{y^r - 1}$, of both terms on the right-hand side of Equation (17) is developed into a convergent series, and their integrals, Φ_1 and Φ_2, may be expressed as follows:

$$\Phi_1 = y - \sum_{n=1}^{\infty} \frac{y^{-(nr-1)}}{nr - 1} \quad \ldots\ldots\ldots\ldots\ldots\ldots.(21a)$$

and,

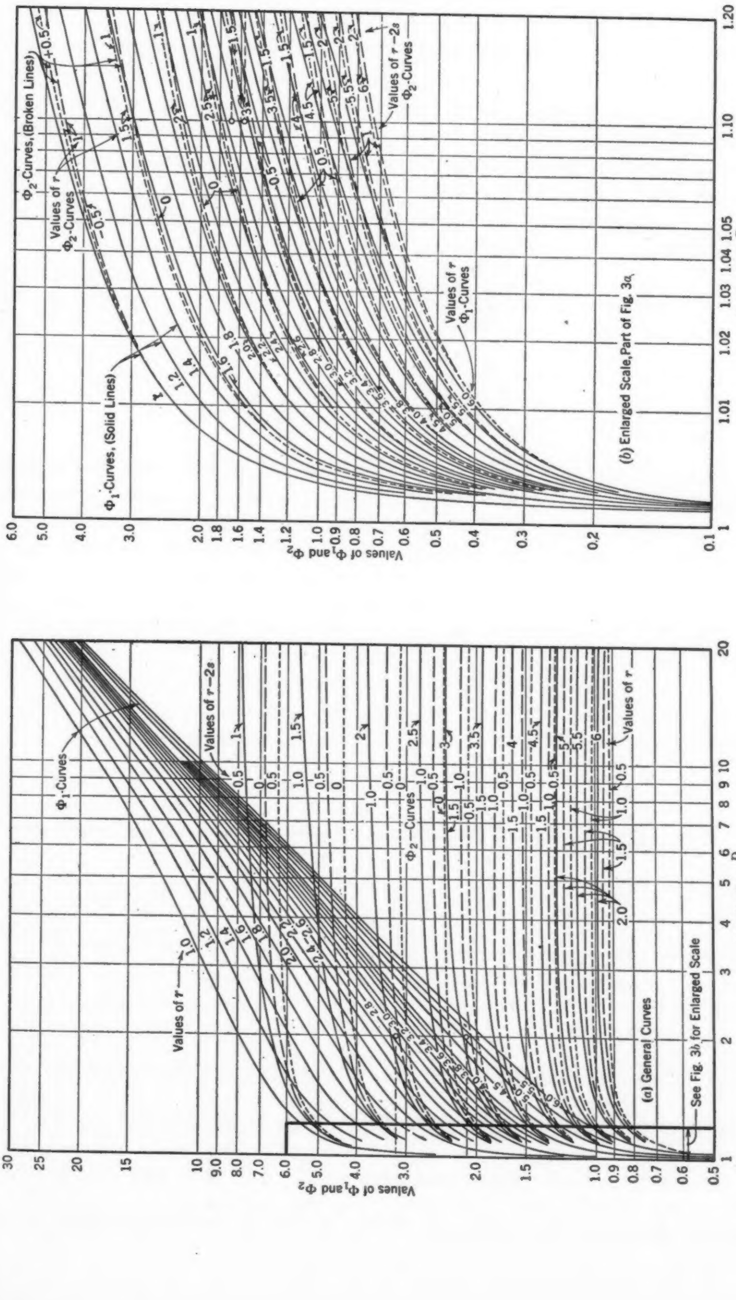


FIG. 3.—BACK-WATER CURVES; VALUES OF Φ_1 AND Φ_2 FOR SUBSTITUTION IN EQUATION (18).

$$\Phi_2 = -\sum_{n=1}^{\infty} \frac{y^{-(nr+2s)}}{nr + 2\,s} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (21b)$$

However, in actual numerical calculation, when y is nearly equal to 1, the rate at which the series converges is very slow. When both r and s are integers or possess special values, Equations (21) can be expressed in a finite number of terms, and in preparing an actual numerical table, Φ_1 and Φ_2 are calculated directly for these special cases and the values of the functions for intermediate values of r and s are obtained by interpolation. The results are shown by groups of curves in Fig. 3, in which, for substitution in Equation (18):

$$\Phi_1\left(\frac{D}{D_0}\right) = \int_{1.001}^{y} \frac{y^r}{y^r - 1}\, dy \quad \ldots\ldots\ldots\ldots\ldots\ldots (22a)$$

and,

$$\Phi_2\left(\frac{D}{D_0}\right) = \int_{1.001}^{y} \frac{y^{(r-2s)-1}}{y^r - 1}\, dy \quad \ldots\ldots\ldots\ldots\ldots\ldots (22b)$$

If groups of Φ_1 and Φ_2 curves are constructed by plotting the integrated values thus obtained, the back-water profile applicable to almost any channel can be calculated precisely. Back-water curves published in the past were for a single case of r and s; for instance, in the Grashof-Bresse formula, a broad rectangular section is used so that, with $A = B\,D$, and $P = B$, $s = 1$, and $k = 0$. Since $V_0 = C\,R^{0.5}{}_0\,S^{0.5}{}_0$, the exponent, m, $= 0.5$. Therefore, $s = 1$, and $r = 3$.

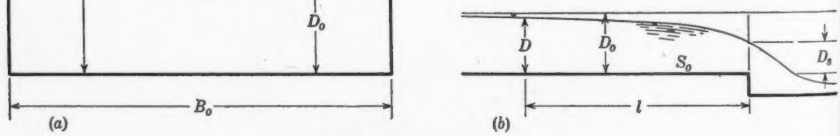


FIG. 4.—EXAMPLES.

In the Tolkmitt formula (Item No. 4, Table 1), a broad parabolic section is used in the Chezy velocity formula so that $m = 0.5$ and $A = \frac{2}{3} BD$; $P = B$; and $R = \frac{2}{3} D$. Consequently, $s = \frac{3}{2}$; $k = 0.5$; and $r = 4$. In the Rühlmann formula (Item No. 3, Table 1) Φ_2 is neglected when $s = 1$ and $r = 3$.

Illustrative Computation for Back-Water.—Referring to Fig. 4(*a*) let $B_0 = 8$ ft; $D_0 = 2$ ft; $D = 2.2$ ft; $D_b = 4$ ft; $S_0 = 1:1\,000$; $C = 50$; $m = 0.7$; $s = 1.0$; $k = 0.38$; $r = 2.87$; and $K = 0.212$. Then, from Fig. 3, selecting proper values of Φ_1 and Φ_2, Equations (22) yield: $\Phi_1\left(\frac{D_b}{D_0}\right) = 3.16$; $\Phi_1\left(\frac{D}{D_0}\right) = 1.68$; $\Phi_2\left(\frac{D_b}{D_0}\right) = 2.12$; and $\Phi_2\left(\frac{D}{D_0}\right) = 1.58$. Substituting these values in Equation (18): $\frac{S_0\,l}{D_0} = 3.16 - 1.68 - 0.212\,(2.12 - 1.58) = 1.365$; and, $l = 2\,730$ ft.

Drop-Down Curve.—In this case the drop-down functions are: $\Psi_1\,(z) = \int \frac{z^{-2}\,dz}{z^r - 1}$; and, $\Psi_2\,(z) = \int \frac{z^{2s-1}}{z^r - 1}\,dz$, and each of them can be expressed as infinite series, thus:

$$\Psi_1\,(z) = -\sum_{n=1}^{\infty} \frac{z^{-(nr+1)}}{nr + 1} \qquad \text{(23a)}$$

and,

$$\Psi_2\,(z) = -\sum_{n=1}^{\infty} \frac{z^{-nr+2s}}{nr - 2s} \qquad \text{(23b)}$$

For integral or special values of r and s, the integration leads to expressions having a finite number of terms, so that if values of the functions are computed by these expressions, the values of the functions for intermediate values of r and s can be obtained easily by interpolation.

The values of Ψ_1 and Ψ_2 are plotted in Figs. 5, 6, and 7, the values in Fig. 6 differing slightly from those in Fig. 5 not only with respect to the values of $\frac{D}{D_0}$, but also with respect to the values of s.

Illustrative Computation for Drop-Down.—Referring to Fig. 4(*b*), let $B_0 = 8$ ft; $D_0 = 4$ ft; $D_s = 2.5$ ft; $D = 3.8$ ft; $Z = \frac{D_0}{D_s} = 1.6$; $z = \frac{D_0}{D} = 1.052$; $S_0 = 1:1\,000$; $m = 0.7$; $s = 1.0$; $k = 0.47$; $r = 2.74$; $C = 60$; and $K = 0.27$. Then, from Figs. 5, 6, and 7: $\Psi_1\left(\frac{D_0}{D_s}\right) = 1.900$; $\Psi_2\left(\frac{D_0}{D_s}\right) = 2.34$; and $\Psi_2\left(\frac{D_0}{D}\right) = 1.45$. Substituting these values in Equation (20): $\frac{S_0\,l}{D_0} = 1.900 - 1.395 - 0.27\,[2.34 - 1.45] = 0.265$; and $l = 1\,060$ ft.

General.—In Figs. 3 to 7, the values of Φ_1, Φ_2, and Ψ_1, Ψ_2, which are shown by groups of curves, are obtained by ordinary integration with respect to a number of sets of values of r and s which furnish directly integrable equations.

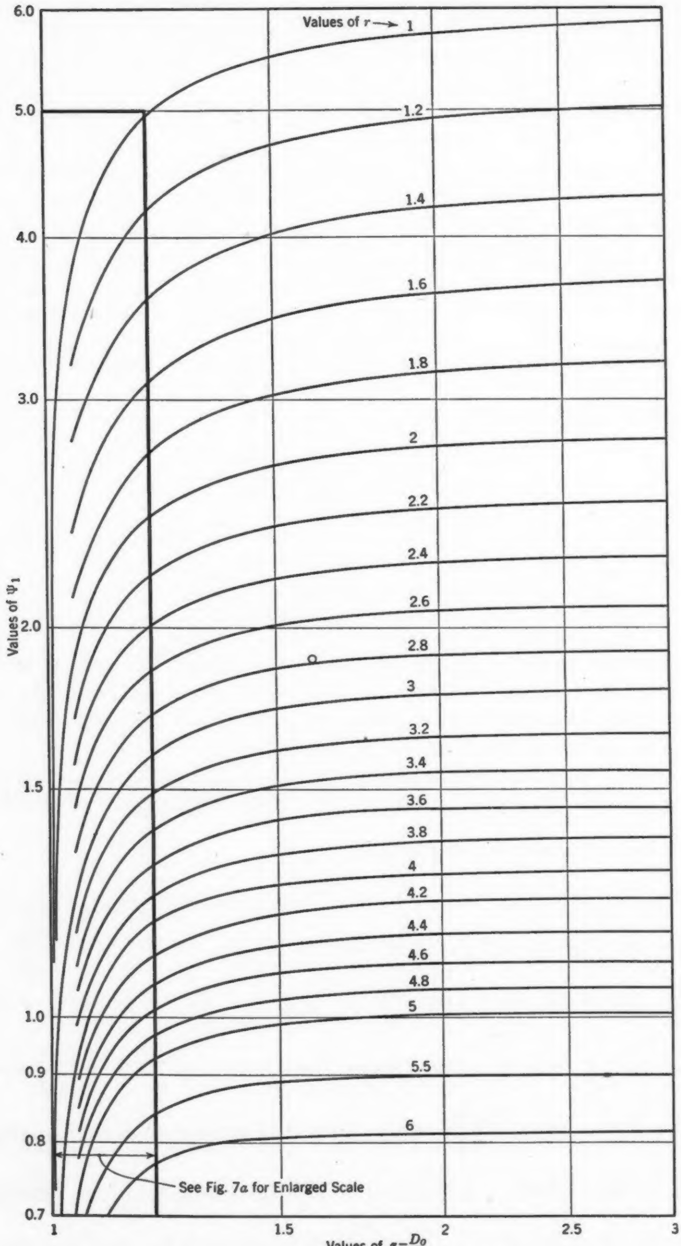


FIG. 5.—DROP-DOWN CURVES; VALUES OF ψ_1 FOR SUBSTITUTION IN EQUATION (20).

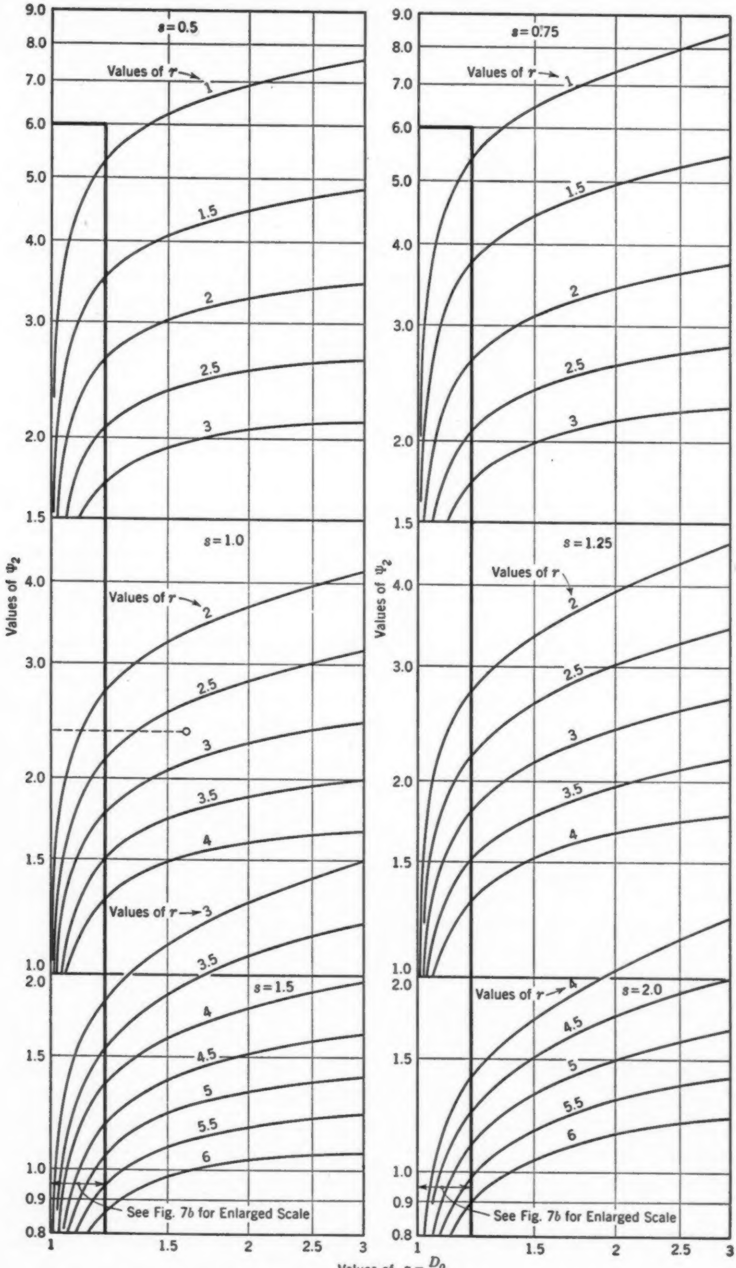


FIG. 6.—DROP-DOWN CURVES; VALUES OF ψ_2 FOR SUBSTITUTION IN EQUATION (20).

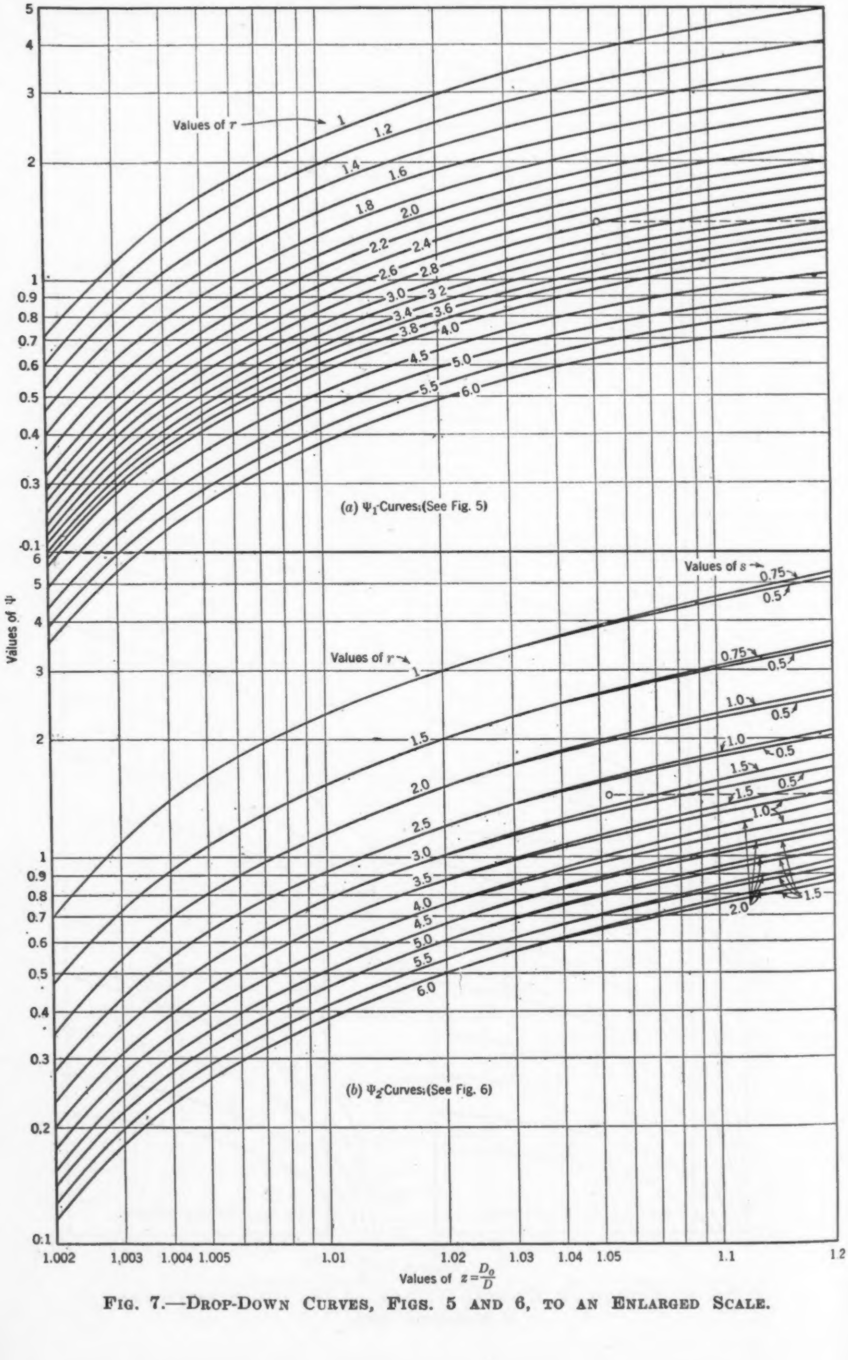


FIG. 7.—DROP-DOWN CURVES, FIGS. 5 AND 6, TO AN ENLARGED SCALE.

Intermediate values of r and s are determined by interpolation. For the back-water curves the direct integration is performed with respect to the following special cases: (*a*) For Φ_1: $r = 1, 2, 3, 4, 5$, and 6; (*b*) $r = 3$ is for Φ_1 in the Grashof-Bresse (Item No. 1, Table 1) and Rühlmann formulas (Item No. 3, Table 1); (*c*) $r = 5$ is for Φ_1 of the Baticle formula (Item No. 7, Table 1); (*d*) values of r and corresponding values of s for integrals of Φ_2 (see Fig. 3) are listed in Table 2; (*e*) $r = 3$, $s = 1.0$ are for Φ_2 of the Grashof-Bresse formula; and (*f*) $r = 4$, $s = 1.5$ are for Φ_2 of the Tolkmitt formula (Item No. 4, Table 1).

TABLE 2.—VALUES OF r, WITH CORRESPONDING VALUES OF s, FOR INTEGRALS OF Φ_2 AND Ψ_2

Values of r	VALUES OF s, FOR:		Values of r	Values of s for Φ_2 and Ψ_2	Values of r	Values of s for Φ_2 and Ψ_2	Values of r	VALUES OF s, FOR:	
	Φ_2	Ψ_2						Φ_2	Ψ_2
(1)	(2)	(3)	(1)	(2)	(1)	(2)	(1)	(2)	(3)
1.0	0.25 0.50 0.75	0.50 0.75 1.00	2.0	0.50 0.75 1.00	3.0	0.75 1.00 1.25 1.50	5.0	1.50 2.00	1.50 2.00
1.5	0.25 0.50 0.75	....	2.5	0.75 1.00 1.25	4.0	1 00 1.50 2.00	6.0	2.00	1.50 2.00 ....

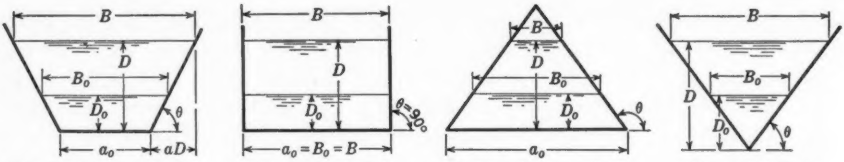
In the drop-down the direct integration is performed with respect to the following special cases: (*g*) For Ψ_1: $r = 1, 2, 3, 3.4, 3.5, 4, 5$, and 6; (*h*) $r = 3$ is for Ψ_1 of the Grashof-Bresse and Rühlmann formulas; (*i*) $r = 3.4$ is for the Kozeny formula (Item No. 8, Table 1); (*j*) $r = 3.5$ is for the Schaffernak formula (Item No. 5, Table 1); (*k*) $r = 4$ is for the Tolkmitt formula; and, (*l*) values of r, with corresponding values of s, for integrals of Ψ_2 are listed in Table 2, in which the combination, $r = 3$ and $s = 1.0$ is the value of Ψ_2 for the Grashof-Bresse formula. Direct integration is possible for seven forms of the integral, Φ_1, twenty-two forms of the integral, Φ_2, eight forms of the integral, Ψ_1, and seventeen forms of the integral, Ψ_2. The integrations were performed by Reidi Ito, River Engineer of the Department of Home Affairs, Japan.

6.—HYDRAULIC AREA, WETTED PERIMETER, AND HYDRAULIC RADIUS

Various shapes of sections which are commonly used for channels, however complicated they may be, can be approximated accurately enough by a combination of straight line, parabolic, and circular curves. Then, establishing the mathematical relation of A, P, and R, or s and k, to the depth of water for these three cases, the factors, s and r, may be obtained for a required depth of water in the case of an irregular section by curves or by direct calculation.

(*a*) *Prismatic Section.*—In the cases shown in Fig. 8 the hydraulic area, A, may be expressed in general terms by:

$$A = a_0 D + a D^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (24)$$



(*a*) Trapezoidal; $a = \cot\theta > 0$ (*b*) Rectangular; $s = 1.0$ and $a = 0$ (*c*) Triangular; $a = \cot\theta < 0$ (*d*) Inverted Triangle; $s = 2.0$ and $k = 1.0$

FIG. 8.—REGULAR CHANNEL CROSS-SECTIONS.

in which $a = \cot\theta$; θ = the angle made by the slope of the bank and the extension of the base line of the channel; and the values of a_0 are as shown in Table 3.

TABLE 3.—VALUES OF a_0 AND θ IN EQUATION (24)

Description	Trapezoidal section (Fig. 8 (*a*))	Rectangular section (Fig. 8 (*b*))	Triangular section (Fig. 8 (*c*))	Inverted triangle (Fig. 8 (*d*))
a_o $\cot\theta$	a_o +	$B = B_o$ 0	a_o −	0 +

From Equation (24):

$$\frac{A}{A_0} = \frac{a_0\,D + a\,D^2}{a_0\,D_0 + a\,D_0^2} = \frac{y + \frac{a}{\mu}\,y^2}{1 + \frac{a}{\mu}} = y^s \qquad (25)$$

in which $y = \frac{D}{D_0}$; $\mu = \frac{a_0}{D_0}$; and $\frac{a}{\mu} = \frac{D_0}{a_0}\cot\theta$.

The expression involving the wetted perimeter, P, is written as:

$$P = a_0 + 2\,D\,\sqrt{1 + a^2} \qquad (26)$$

from which,

$$\frac{P}{P_0} = \frac{a_0 + 2\,D\sqrt{1 + a^2}}{a_0 + 2\,D_0\sqrt{1 + a^2}} = \frac{1 + 2\,y\,\sqrt{\frac{1 + a^2}{\mu^2}}}{1 + 2\,\sqrt{\frac{1 + a^2}{\mu^2}}} = y^k \qquad (27)$$

Then, s and k, the exponents to y which depend on the ratios, $\frac{D}{D_0}$, are computed for various values of $\frac{a}{\mu}$ and $\frac{1 + a^2}{\mu^2}$, with y varying between 0.1 and 20. The curves are plotted as shown in Fig. 9. For example, referring to Fig. 8(*b*), let $a_0 = 8$ ft; $D_0 = 2$ ft; $D_b = 4$ ft; $D = 2.2$ ft; values of y at two sections considered are 2.0 and 1.1; $\mu = 4$; $\frac{a}{\mu} = 0$; $\frac{1 + a^2}{\mu^2} = \frac{1 + 0^2}{4^2}$

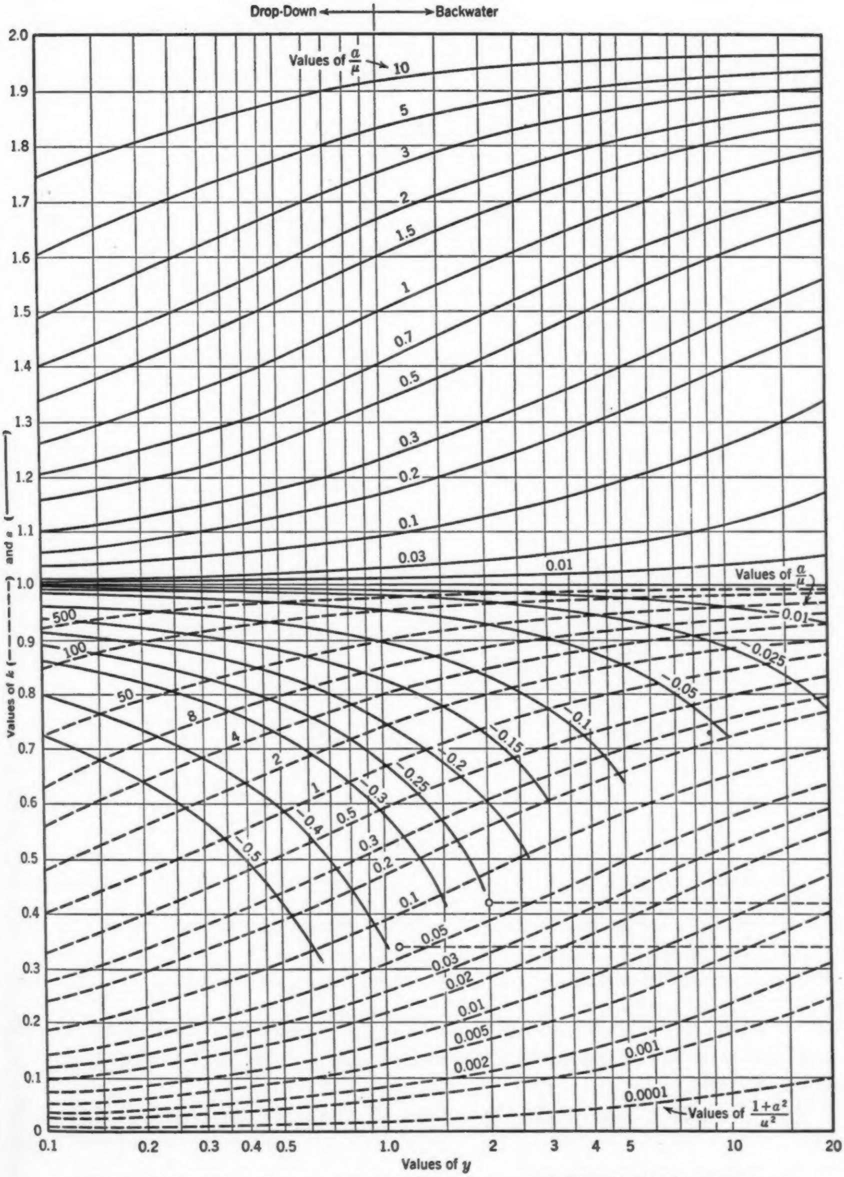


FIG. 9.—VALUES OF k AND s FOR CHANNELS WITH PRISMATIC CROSS-SECTIONS.

$= 0.0625$; and $s = 1.0$, a constant. From Fig. 9, $k = 0.34$ to 0.42, a mean value of 0.38.

(*b*) *Parabolic Section.*—In the case of the parabolic section shown in Fig. 10(*a*), $A = \frac{2}{3} BD$; $A_o = \frac{2}{3} B_o D_o$; and $B = B_o \sqrt{\frac{D}{D_o}}$. Therefore, $\frac{A}{A_o} = y^{\frac{3}{2}} = y^s$; and, $s = 1.5$, a constant. Finally,

$$P = 2\, D_o \left[\sqrt{y \left(y + \frac{B^2_o}{16\, D^2_o} \right)} + \frac{B^2_o}{16\, D^2_o} \log_e \frac{\sqrt{y} + \sqrt{y + \frac{B^2_o}{16\, D^2_o}}}{\frac{B_o}{4\, D_o}} \right]$$

and,

$$\frac{P}{P_o} = \frac{\sqrt{y \left(y + \frac{1}{16} \frac{B^2_o}{D^2_o} \right)} + \frac{1}{16} \frac{B^2_o}{D^2_o} \cdot \log_e \frac{\sqrt{y} + \sqrt{y + \frac{B^2_o}{16\, D^2_o}}}{\frac{B_o}{4\, D_o}}}{\sqrt{1 + \frac{1}{16} \frac{B^2_o}{D^2_o}} + \frac{1}{16} \frac{B^2_o}{D^2_o} \log_e \frac{4\, D_o}{B_o} \left[1 + \sqrt{1 + \frac{B^2_o}{16\, D^2_o}} \right]} = y^k \quad \text{..(28)}$$

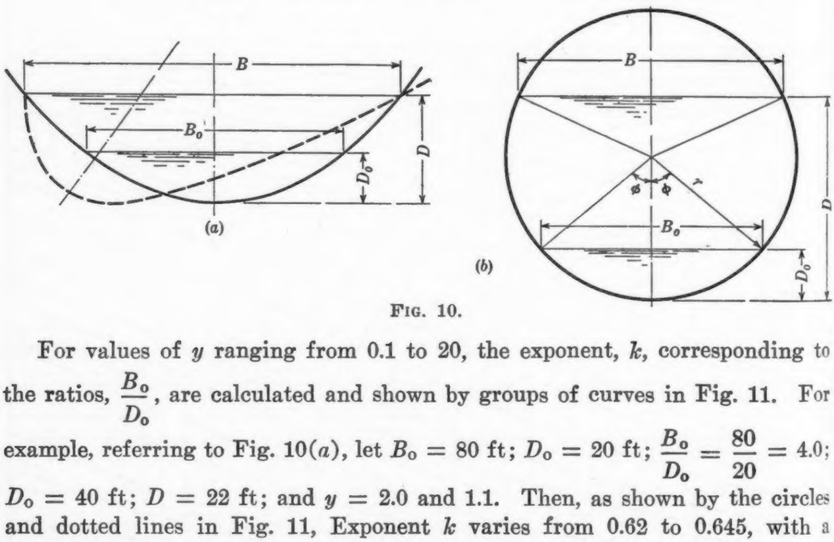


FIG. 10.

For values of y ranging from 0.1 to 20, the exponent, k, corresponding to the ratios, $\frac{B_o}{D_o}$, are calculated and shown by groups of curves in Fig. 11. For example, referring to Fig. 10(*a*), let $B_o = 80$ ft; $D_o = 20$ ft; $\frac{B_o}{D_o} = \frac{80}{20} = 4.0$; $D_o = 40$ ft; $D = 22$ ft; and $y = 2.0$ and 1.1. Then, as shown by the circles and dotted lines in Fig. 11, Exponent k varies from 0.62 to 0.645, with a mean of 0.632.

(*c*) *Circular Section.*—In the case of the circular section shown in Fig. 10(*b*), r is the radius of the circle; ϕ equals one-half the central angle of the wetted perimeter; and

$$A = r^2 \cos^{-1}\left(\frac{r - D}{r}\right) + (D - r) \sqrt{2\, Dr - D^2} \quad \text{.........(29)}$$

so that,

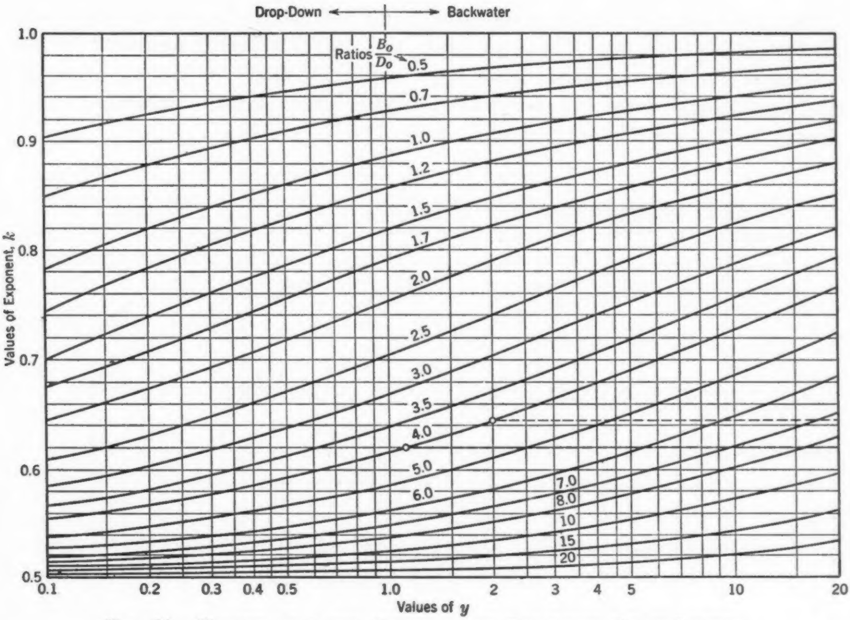


FIG. 11.—VALUES OF k FOR CHANNELS OF PARABOLIC CROSS-SECTION.

$$\frac{A}{A_o} = \frac{\cos^{-1}(1-x) + (x-1)\sqrt{2\,x - x^2}}{\cos^{-1}(1-x_o) + (x_o - 1)\sqrt{2\,x_o - x^2_o}}\,y^s \ldots\ldots\ldots(30)$$

in which, $x = \frac{D}{r}$; $x_o = \frac{D_o}{r}$; $y = \frac{D}{D_o} = \frac{x}{x_o}$; and,

$$P = 2\,\phi\,r = 2\,r\cos^{-1}\left(\frac{r-D}{r}\right) \ldots\ldots\ldots\ldots\ldots(31)$$

Consequently,

$$\frac{P}{P_o} = \frac{\cos^{-1}(1-x)}{\cos^{-1}(1-x_o)} = y^k \ldots\ldots\ldots\ldots\ldots\ldots(32)$$

For values of y ranging from 0.1 to 20, the exponents, s and k, corresponding to the ratios of $x_o = \frac{D_o}{r}$ are calculated and shown in Fig. 12. For example, referring to Fig. 10(b), let $r = 9$ ft; $D_o = 4.5$ ft; $D_b = 13.5$ ft; $D = 5.4$ ft; $x_o = \frac{D_o}{r} = 0.5$; $Y = \frac{D_b}{D_o} = 3.0$; and $y = \frac{D}{D_o} = 1.2$. Then, as shown by the circles in Fig. 12, Exponent s lies between 1.28 and 1.40 (mean, 1.34) and Exponent k lies between 0.55 and 0.63 (mean, 0.59).

(d) *Error Involved in Assuming That Section Indices, s and k, Are Constant.*—Generally, the values of s and k in Equations (13a) and (13b) in any section differ slightly with changes in the depth of water, D, but, in this

paper, the average value for the range in which D changes is used in order to facilitate the integration. To study the degree of error produced by this assumption, comparisons are made between two examples in which rectangular sections are used for the back-water experiments and the results obtained from the Dupuit-Masoni formula.

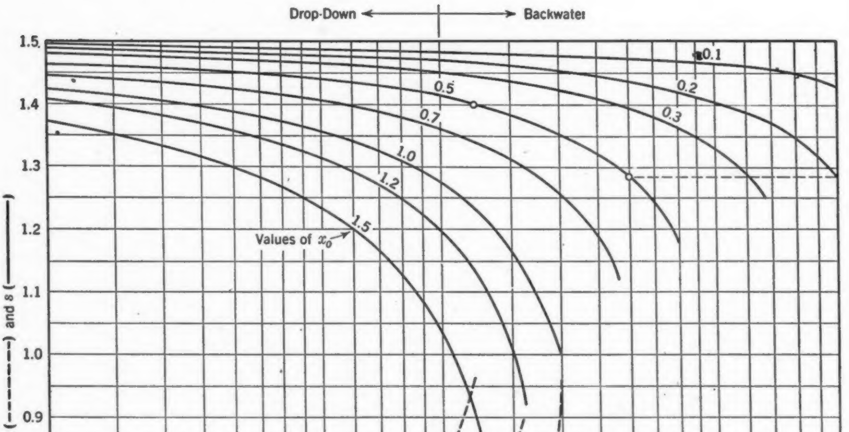


FIG. 12.—VALUES OF s AND k FOR CHANNELS OF CIRCULAR CROSS-SECTION.

Referring to Fig. 4(a), for Example 1, let $\frac{B_0}{D_0} = 2.5$; $Y = \frac{D_b}{D_0} = 2.0$; $y = \frac{D}{D_0} = 1.01$; and $k = 0.54$ to 0.46 (mean assumed as 0.5). For Example 2, let $\frac{B_0}{D_0} = 3.0$; $Y = 5.0$; $y = 1.01$; and $k = 0.60$ to 0.40 (mean assumed as 0.5). For convenience in comparing them with the Dupuit-Masoni formula (Item No. 2, Table 1), let $V = C\ R^m\ S^{0.5}$ and $m = 0.5$. Then, $r = 2\ s + 2\ m\ (s - k) = 2.5$;

$$K = \alpha\, s\, \frac{V^2_0}{g\, D_0} = \beta\, \frac{R_0}{D_0} = \beta\, \frac{B_0}{B_0 + 2\, D_0} \quad \text{............} (33)$$

and,

$$\beta = \alpha\, \frac{C^2\, S_0}{g} = \alpha\, \frac{V^2_0}{g\, R_0} \quad \text{....................} (34)$$

If k is considered a variable, the Dupuit-Masoni formula results, and the relation between β and K are as shown in Table 4.

TABLE 4.—RELATION BETWEEN K AND β, EQUATIONS (33) AND (34)

Values of:	EXAMPLE; 1 $Y = 2.0$		EXAMPLE 2; $Y = 5.0$	
	$\beta = 0.3$	$\beta = 0.9$	$\beta = 0.3$	$\beta = 0.9$
K	0.17	0.50	0.18	0.54

If the values of the ratio, $\frac{D}{D_0}$, are compared with the numerical values of $\frac{S_0\, l}{D_0}$ (which are proportional to the distance, l, measured from the weir to various sections up stream from it), errors between the Dupuit-Masoni formula and Equation (17) will be as indicated by Table 5.

TABLE 5.—EFFECT OF ASSUMING AN AVERAGE VALUE OF K; COMPARISON OF THE RATIO, $\frac{S_0 l}{D_0}$

Ratio, $y = \frac{D}{D_0}$	EXAMPLE 1.— $Y=2.0$						EXAMPLE 2.— $Y=5.0$					
	$\beta=0.3$ and $K=0.17$			$\beta=0.9$ and $K=0.5$			$\beta=0.30$ and $K=0.18$			$\beta=0.90$ and $K=0.54$		
	Using exact value of k	Using mean value of k	Percentage error	Using exact value of k	Using mean value of k	Percentage error	Using exact value of k	Using mean value of k	Percentage error	Using exact value of k	Using mean value of k	Percentage error
(1)	(2)	(3)	(4)	(2)	(3)	(4)	(2)	(3)	(4)	(2)	(3)	(4)
2.0	......	......	......	......	......	......	3.1800	3.1724	−0.24	3.1381	3.1306	−0.24
1.7	0.3742	0.3732	−0.27	0.3539	0.3527	−0.34	......	......	......	......	......	......
1.5	0.6526	0.6517	−0.14	0.6086	0.6074	−0.20	3.8261	3.8224	−0.10	3.7365	3.7327	−0.10
1.3	0.9849	0.9853	+0.04	0.8972	0.8968	−0.04	......	......	......	......	......	......
1.2	1.1990	1.2016	+0.22	1.0710	1.0723	+0.12	......	......	......	......	......	......
1.1	1.5049	1.5123	+0.49	1.3005	1.3052	+0.36	4.6624	4.6764	+0.30	4.4009	4.4110	+0.23
1.05	1.7710	1.7828	+0.67	1.4835	1.4911	+0.51	4.9211	4.9435	+0.46	4.5710	4.5867	+0.34
1.01	2.3274	2.3496	−0.95	1.8361	1.8498	+0.75	5.4608	5.5020	+0.75	4.8939	4.9205	+0.54

Evidently, errors produced by integration, using average values of k which are considered as constants, are insignificant, being usually within 1 per cent. Generally, the errors tend to increase in the up-stream direction where the change in the depth of water is slight. If it is desired to find the shape of the back-water curve more accurately for theoretical reasons, groups of curves similar to those in Fig. 3 are drawn by using the average values of k only for the integration, r being determined by using actual values of k for the depths of water in a given channel. It is possible to find the values of ϕ_1 and ϕ_2 from the curves corresponding to r and s thus obtained; but (except when the weir is unusually high) the calculation is simplified considerably if the average values of k are used.

7.—Calculated Examples of the Back-Water and Drop-Down in Prismatic Channels

In order to demonstrate the method of computing the back-water and drop-down curves presented in this paper, two general examples are offered.

Example (A).—Back-Water: Case of Trapezoidal Channel.—Referring to Fig. 8(a), let the side slope = 1 on 1; $D_b = 30$ ft; $D_o = 6$ ft; $\theta = 45°$; $a_o = 30$ ft; $S_o = 1{:}1\,600$; $A_o = 6 \times 30 + 6 \times 6 = 216$ sq ft; and $R_o = \dfrac{216}{30 + 2 \times 6\sqrt{2}} = 4.6$ ft. It is necesary to use a velocity equation in the form of Equation (2) in which C remains constant irrespective of R. Now, if the Manning formula is used (Equation (2), with $m = \frac{2}{3}$ and $p = \frac{1}{2}$), taking Kutter's roughness coefficient as $n = 0.020$, the mean velocity is: $V_o = \dfrac{1.486}{0.02} \times 4.6^{\frac{2}{3}} \times \dfrac{1}{\sqrt{1\,600}} = 5.15$ ft per sec. Therefore, $Q = V_o\,A_o = 5.15 \times 216 = 1\,112.4$ cu ft per sec.

If the depth of water is increased to $D_b = 30$ ft by a weir; then $y = \dfrac{D_b}{D_o} = \dfrac{30}{6} = 5$. By Example (1) of Article 6, if $\theta = 45°$ then, $a = \cot\theta = 1$; $\mu = \dfrac{a_o}{D_o} = \dfrac{30}{6} = 5$; $\dfrac{a}{\mu} = \dfrac{1}{5}$; and, $\dfrac{1 + a^2}{\mu^2} = \dfrac{1 + 1}{25} = 0.08$. It is required to find l, the distance from the weir to the section where the depth of water is 18 ft; thus, $y = \dfrac{D}{D_o} = \dfrac{18}{6} = 3$; and $m = \dfrac{2}{3}$. Hence, from Fig. 9 may be selected the values listed in Table 6(a); and with

TABLE 6.—Illustrative Example; Values Read from Curves

Symbol	Example (A): For depth, $D = 18$ feet	Example (A): For depth, $D_b = 30$ feet	Symbol	Example (B): For depth, $D = 12$ feet $\left(\frac{D_o}{D} = 1.25\right)$	Example (B): For depth, $D_s = 10$ feet $\left(\frac{D_o}{D_s} = 1.5\right)$
(1)	(2)	(3)	(4)	(5)	(6)
(a) Values from Fig. 9			(c) Values from Fig. 9		
y	3	5	$\frac{a}{\mu}$	0	0
s	1.31	1.36	$\frac{1+a^2}{\mu_2}$	0.25	0.25
k	0.51	0.56	s	1.0	1.0
$r-2s$	1.07	1.07	k	0.47	0.45
r	3.69	3.79	$r-2s$†	0.707	0.733
K	0.194*	....	r	2.707	2.733
(b) Values from Fig. 3			(d) Values from Figs. 5 and 6		
Φ_1	3.62	5.68	Ψ_1	1.82	1.92
Φ_2	1.80	1.80	Ψ_2	2.08	2.28

* Equation (15). † Equals $2m\,(s-k)$.

these, the integrals, Φ_1 and Φ_2, listed in Table 6(*b*) may be selected from Fig. 3. Substitution in Equation (18) then yields: $\frac{S_0\,l}{D_0} = 5.68 - 3.62 - 0.194\,(1.80 - 1.80) = 2.06$; and, therefore, $l = 2.06 \times \frac{D_0}{S_0} = 2.06 \times 6 \times 1\,600 = 19\,776$ ft. In this case the last quantity of Equation (18) (which is multiplied by K) is insignificant, but it can not be overlooked when $\frac{D_b}{D_0} < 2$.

Example (B).—Drop-Down: Case of Rectangular Channel.—Referring to Fig. 4(*b*), let $D_0 = 15$ ft; $D_s = 10$ ft; $B_0 = a_0 = 30$ ft; $S_0 = 1{:}2\,500$; $n = 0.02$; $C = \frac{1.486}{n} = 74.3$; $A_0 = 30 \times 15 = 450$ sq ft; $R_0 = \frac{450}{30 + 2 \times 15} = 7.5$ ft; and, by Equation (2) with $m = \frac{2}{3}$ and $p = \frac{1}{2}$ (the Manning formula), $V_0 = 74.3 \times 7.5^{\frac{2}{3}} \times \frac{1}{\sqrt{2\,500}} = 5.69$ ft per sec. Consequently, $Q = 450 \times 5.69 = 2\,560$ cu ft per sec; and $K = \alpha\, s\, \frac{V^2_0}{gD_0} = \frac{10}{9} \times 1 \times \frac{5.69^2}{32.2 \times 15} = 0.074$.

If D_s is equal to 10 ft (that is, if it is nearly equal to the critical depth, at the cross-section over the fall): $\mu = \frac{a_0}{D_0} = 2$; $\alpha = \cot\theta = 0$: $\frac{a}{\mu} = 0$; $s = 1$; and, $\frac{1 + a^2}{\mu^2} = 0.25$. (See Figs. 4(*b*) and 8(*b*).)

Hence, to find l, the distance from the fall in the channel to the section where $D = 12$ ft: $y_0 = \frac{D_s}{D_0} = \frac{10}{15} = \frac{2}{3}$; and $y = \frac{D}{D_0} = \frac{12}{15} = 0.8$. From Fig. 9 may be selected values listed in Table 6(*c*); and, with these, the integrals, Ψ_1, and Ψ_2, listed in Table 6(*d*) may be selected from Figs. 5 and 6.

Substitution in Equation (20) then yields $\frac{S_0\,l}{D_0} = 1.92 - 1.82 - 0.074\,(2.28 - 2.08) = 0.085$; and, therefore, $l = 0.085 \times 2\,500 \times 15 = 3\,190$ (ft).

8.—Experimental Flumes

In general, the deciding factors in establishing the shape of the back-water curve are the shape of the channel cross-section and the degree of roughness. The former is the more important and the inadequacy of previous formulas is due, in large measure, to ignoring its relative effect.

In the experiments described herein, both the cross-sectional shape and the surface roughness of the experimental channels were varied. Moreover, due to the shortness of the effective length of the flume in this experiment, it was impossible to determine the height of the water surface covering the total length of back-water limits by a single experiment. For example, when $Y = \frac{D_b}{D_0} = 2$, the back-water profile is equivalent to a curve constructed by

connecting many back-water curves plotted in the ranges from: $\frac{D_b}{D_0} = 2$ to $\frac{D_b}{D_0} = 1.6$; 1.6 to 1.2; 1.2 to 1.05; etc. Hence, in the experiments reported herein, the entire back-water curve was traced by connecting several back-water curves. The curve thus obtained contains from three to nine experimental values which indicates that a considerable number of measurements is required for channels of easy slope, S_0. On the other hand, if S_0 is too steep, the velocity of the water becomes very great and the flow unsteady; in extreme cases, the water shoots ahead so violently that the bed slope up stream from the weir is necessarily fixed at 1:500 and that of the drop-down at from 1:500 to 1:2 000. In order to study the influence of roughness, two kinds of sections were devised for each type of channel section: (1) A smooth flume in which the inner surfaces were planed carefully; and (2), a rough flume in which the inner surfaces were lined with $\frac{3}{8}$-in. mesh made of No. 20 galvanized wire. In addition, the outer surfaces of the flume were submerged in water to avoid deforming the flume by alternate wetting and drying. When the flow was variable, the distance, l (from the weir to the section up stream, where the depth of water is D), was measured, and the back-water and drop-down are represented by the relative values of $\frac{S_0\, l}{D_0}$.

The typical sections of the channels studied in the writer's laboratory are shown in Fig. 13, the dimensions being as indicated. For measuring the discharge a common weir was used; and the height of the water surface was read with an ordinary engineers' level and a needle-gauge readable to the nearest 0.1 mm (0.00394 in.).

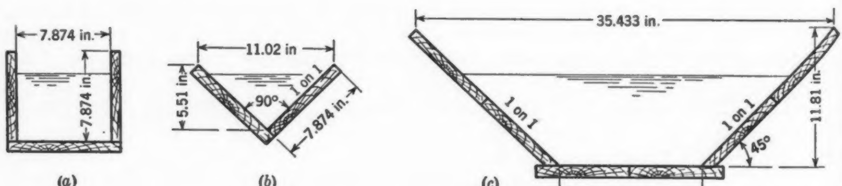


FIG. 13.

Water depths were observed at 0.5-m (16.41-in.) intervals for rectangular and triangular channels and 1.00-m (3.28-ft) intervals for trapezodial flumes. However, readings are taken at 0.2-m (0.656-ft) intervals to measure the elevation of the water surface at important sections, such as the lowest point of the drop-down, where the depth of water changes abruptly.

At the down-stream end of each channel a brass, double-rack gate regulated the water level so that it was nearly the same at the down-stream end in one set of measurements as it was at the up-stream end, in the preceding set. The regulation was effected by means of slits.

9.—The Measurement of Back-Water

Smooth Rectangular Channels.—Referring to Figs. 8(*b*) and 13(*a*), let: $B_0 = 7.874$ in.; $S_0 = 1{:}500$; $D_0 = 1.9685$ in.; $D_b = 3.937$ in.; $V_0 = 1.61$ ft per sec; and $Y = \frac{D_b}{D_0} = 2.0$.

Using Manning's formula for uniform flow, with various slopes, and assuming C to be constant, the writer found by actual measurements that the value of m was also nearly a constant, and was equal to 0.65.

The effective length of the channel was 11 m (36.09 ft). The dam at the lowest down-stream section was adjusted so that $D_b = 2\ D_0 = 3.937$ in., and the elevation of the water surface was measured at 0.5-m intervals (1.64 ft). Then the dam was re-adjusted to make $D_b = 80$ mm (3.15 in.) and the water surface elevations were measured again. This process was repeated until four series of measurements had been made for different values of D_b; then, each set was plotted and the back-water curve from 3.937 in. to 2.087 in. (the depths of water above the weir), was measured. The result thus obtained is shown by dots in Fig. 14(*a*); corresponding data pertaining to rough channels are shown.

Computation of the Ratio, $\frac{S_0\, l}{D_0}$.—In a rectangular channel such as Fig. 4(*a*), in which $s = 1.0$, the mean value of k between $Y = 2.0$ and $Y = 1.01$ may be determined as follows: $\frac{(1 + a^2)}{\mu^2} = \frac{1 + 0}{4^2} = \frac{1}{16}$; and, from Fig. 9, $k = 0.38$. Therefore, $r = 2\,s + 2\,m\ (s - k) = 2.00 + 2 \times 0.65\ (1 - 0.38) = 2.81$; $r - 2\,s = 0.81$; and, $K = \alpha\ s\ \frac{V^2_0}{gD_0} = 0.55$.

After reading the values of Φ_1 and Φ_2 for $r = 2.81$ and $r - 2\,s = 0.81$ which correspond to various values of $y = \frac{D}{D_0}$ from Fig. 3, the values of $\frac{S_0\, l}{D_0}$ can be computed by Equation (18) and arranged as in Table 7(*a*). Next, in order to compare the errors in various back-water formulas that have been used heretofore (see Table 1), the values of $\frac{S_0\, l}{D_0}$ computed from these formulas are likewise tabulated, and the back-water curves are drawn, for general comparison, as shown in Fig. 14.

Finally, the errors in the various current formulas are determined by comparing them with the results of the writer's tests. Because of minute surface and stationary waves which exist in the experimental channels, variations of a fraction of a millimeter occur in the test readings. In some cases these variations make it almost impossible to determine the actual error of a formula.

Although the magnitude of the errors in the experimental observations was very small, the effects of these errors on the back-water curves for small channels were quite remarkable. In the preliminary tests the measured values were quite erratic, due to deflections in the channel structure between beam

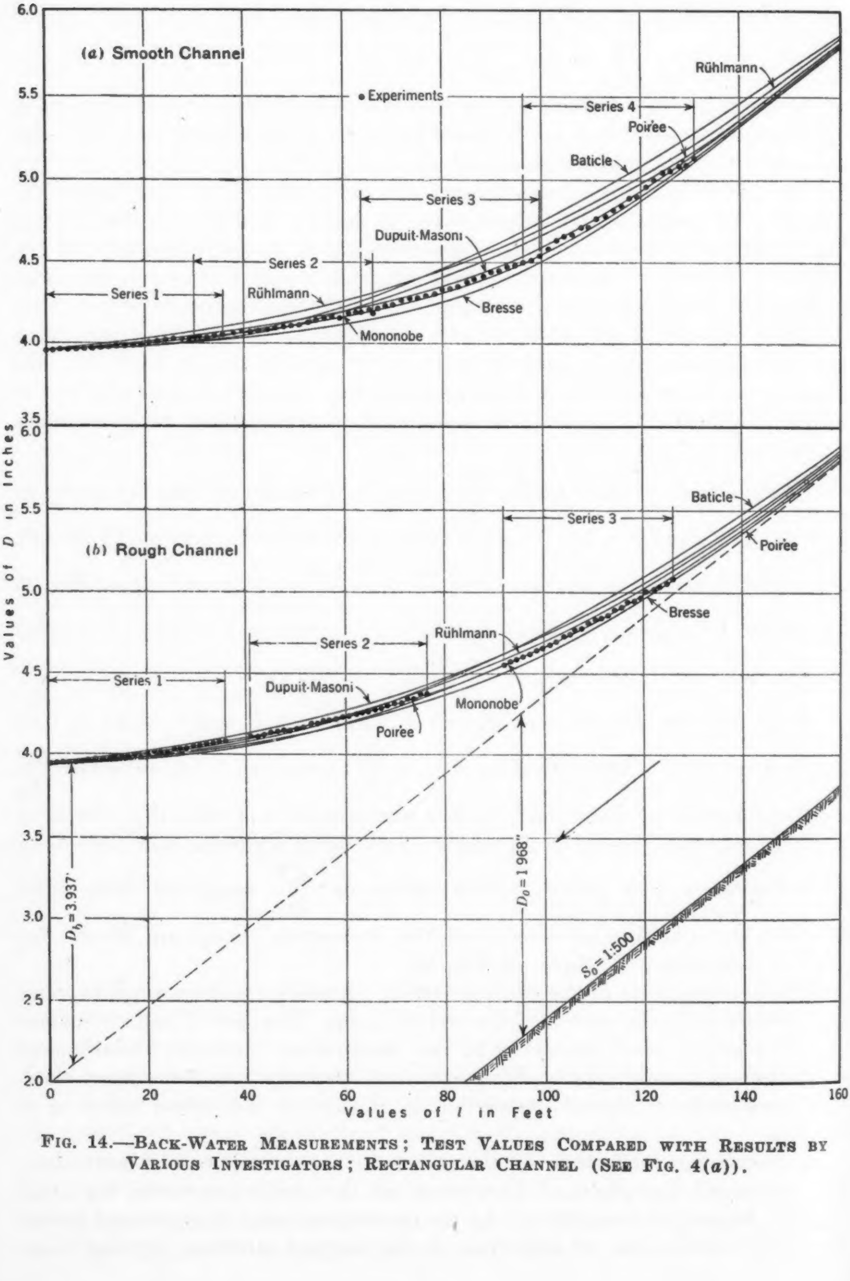


FIG. 14.—BACK-WATER MEASUREMENTS; TEST VALUES COMPARED WITH RESULTS BY VARIOUS INVESTIGATORS; RECTANGULAR CHANNEL (SEE FIG. 4(a)).

TABLE 7.—BACK-WATER MEASUREMENTS; COMPARISON OF THE RATIO, $\frac{S_0\, l}{D_0}$, DETERMINED BY EXPERIMENT AND BY FORMULAS

Formulas	RATIOS, $\frac{S_0 l}{D_0}$, FOR THE FOLLOWING VALUES OF $y = \frac{D}{D_0}$:										
	1.9	1.8	1.7	1.6	1.5	1.4	1.3	1.2	1.1	1.05	1.01
	AND FOR THE CORRESPONDING VALUES OF D, IN INCHES:										
	3.740	3.543	3.347	3.150	2.953	2.756	2.559	2.362	2.165	2.070	1.990
	(*a*) SMOOTH RECTANGULAR CHANNEL										
Laboratory test	0.111	0.222	0.344	0.461	0.581	0.714	0.853	1.020	1.200	*	*
Mononobe	0.110	0.224	0.340	0.459	0.584	0.707	0.844	0.992	1.194	1.333	1.655
Dupuit-Masoni	0.113	0.227	0.345	0.465	0.591	0.724	0.867	1.030	1.242	1.409	1.724
Bresse	0.107	0.215	0.326	0.439	0.555	0.675	0.808	0.956	1.147	1.299	1.569
Rühlmann	0.116	0.234	0.359	0.486	0.623	0.772	0.941	1.148	1.449	1.714	2.277
Schaffernak	0.112	0.228	0.344	0.465	0.590	0.727	0.855	1.070	1.328	1.558	*
Ehrenberger	0.110	0.216	0.330	0.446	0.570	0.701	0.849	1.021	1.270	1.466	1.766
Baticle	0.102	0.210	0.329	0.458	0.600	0.757	0.954	1.198	1.576	1.943	2.690
Poirée	0.103	0.211	0.327	0.451	0.586	0.735	0.905	1.106	1.368	1.553	1.800
Tolkmitt	0.101	0.203	0.306	0.409	0.512	0.618	0.726	0.840	0.966	1.047	1.158
	(*b*) ROUGH RECTANGULAR CHANNEL										
Laboratory test	0.113	0.231	0.355	0.485	0.619	0.764	0.922	1.106	1.388	*	*
Mononobe	0.115	0.233	0.355	0.482	0.615	0.759	0.919	1.110	1.384	1.627	2.134
Dupuit-Masoni	0.119	0.241	0.368	0.500	0.640	0.792	0.963	1.166	1.464	1.722	2.256
Bresse	0.113	0.229	0.348	0.472	0.603	0.740	0.903	1.092	1.361	1.595	2.065
Rühlmann	0.116	0.234	0.359	0.486	0.623	0.772	0.941	1.148	1.449	1.714	2.277
Schaffernak	0.112	0.228	0.344	0.465	0.590	0.727	0.855	1.070	1.328	1.558	*
Ehrenberger	0.110	0.216	0.330	0.446	0.570	0.701	0.849	1.021	1.270	1.466	1.766
Baticle	0.102	0.210	0.329	0.458	0.600	0.757	0.954	1.198	1.576	1.943	2.690
Poirée	0.103	0.211	0.327	0.451	0.586	0.735	0.905	1.106	1.368	1.553	1.800
Tolkmitt	0.105	0.213	0.322	0.434	0.550	0.670	0.805	0.959	1.166	1.333	1.669
	(*c*) SMOOTH, RIGHT-TRIANGULAR CHANNEL										
Laboratory test	0.100	0.201	0.302	0.404	0.510	0.615	0.725	0.848	0.986	1.102	*
Mononobe	0.100	0.201	0.303	0.405	0.512	0.622	0.736	0.854	1.007	1.118	1.390
Baticle	0.104	0.209	0.315	0.424	0.537	0.655	0.784	0.934	1.139	1.309	1.655
Tolkmitt	0.105	0.213	0.322	0.434	0.550	0.672	0.805	0.960	1.166	1.335	1.673
Bresse	0.113	0.229	0.348	0.473	0.604	0.740	0.903	1.093	1.362	1.547	2.073
Rühlmann	0.116	0.234	0.359	0.486	0.623	0.772	0.941	1.148	1.449	1.714	2.277
Schaffernak	0.112	0.228	0.344	0.465	0.590	0.727	0.885	1.070	1.328	1.558	*
Ehrenberger	0.111	0.216	0.330	0.446	0.570	0.701	0.849	1.021	1.270	1.466	1.766
Poirée	0.103	0.211	0.327	0.451	0.586	0.735	0.905	1.106	1.368	1.553	1.800
	(*d*) ROUGH, TRAPEZOIDAL CHANNEL										
Values of D, in inches †	7.480	7.087	6.693	6.299	5.906	5.512	5.118	4.724	4.331	4.130	3.980
Laboratory test	0.105	0.212	0.322	0.436	0.552	0.678	0.814	0.992	1.194	1.370	*
Mononobe	0.110	0.217	0.327	0.436	0.554	0.681	0.818	0.979	1.204	1.373	1.723
Tolkmitt	0.107	0.211	0.320	0.429	0.544	0.663	0.792	0.939	1.132	1.285	1.578
Bresse	0.112	0.226	0.345	0.467	0.595	0.729	0.887	1.070	1.326	1.499	1.991
Ehrenberger	0.111	0.216	0.330	0.446	0.570	0.701	0.849	1.021	1.270	1.466	1.766
Rühlmann	0.116	0.234	0.359	0.486	0.623	0.772	0.941	1.148	1.449	1.714	2.277
Schaffernak	0.112	0.227	0.344	0.465	0.589	0.728	0.885	1.070	1.328	1.558	*
Baticle	0.106	0.217	0.332	0.452	0.577	0.712	0.872	1.078	1.374	1.640	2.203
Poirée	0.103	0.211	0.323	0.451	0.586	0.735	0.905	1.106	1.368	1.553	1.800

* Value not determinable. † Different from Tables 7(*a*), 7(*b*), and 7(*c*).

supports. This condition was caused (1) by the increase in the total weight when water was impounded by the weir and absorbed to some extent by the channel; and (2) by complicating factors at the intake and outlet. Subsequently, however, these defects were mostly corrected; the experiments were conducted under almost ideal conditions; and the results coincided fairly well with the writer's calculated values.

For example, comparing the experimental and calculated values of various formulas of back-water (see Fig. 14(*a*)) for the case of a smooth rectangular

flume, the Dupuit-Masoni formula is found to be the most rational, but the values of $\frac{S_0 l}{D_0}$ are a little greater, due to using the Chezy formula in which C is considered as a constant (that is, m being assumed equal to 0.5 in Equation (2)). However, this formula (Item No. 2, Table 1) agrees more nearly with experiments than any of the others.

The Dupuit-Masoni formula was adopted for a study of the relation derived by Bresse, and the section was a broad one. Errors due to these two assumptions are negative in sign and their magnitude is far greater than the positive errors due to assuming a positive value of C. Consequently, the ratios, $\frac{S_0 l}{D_0}$, by the Bresse formula are slightly too small.

The Rühlmann formula (Item No. 3, Table 1) is derived by neglecting the change in velocity head in the Bresse formula. It yields values considerably in excess of those found by the writer, using a smooth channel with a steep slope and a relatively high velocity. This is particularly true toward the up-stream end of the profile where the rise in water surface is slight and, hence, this formula is not applicable when it is desired to find the position of a point which is to be regarded as the upper limit of the back-water.

The Schaffernak formula (Item No. 5, Table 1) is based upon almost the same assumption as the Rühlmann formula. It was found to coincide somewhat better than the Rühlmann formula with experimental results, since m is equal to 0.75, C being regarded as a constant. Values of $\frac{S_0 l}{D_0}$ by this formula are assumed to be considerably greater because the rectangle is assumed to be broad and the velocity head is disregarded.

In applying the Ehrenberger formula (Item No. 6, Table 1), the rectangle is assumed to be broad, which gives rise to negative errors, but those caused by neglecting the velocity head tend to offset them so that, in general, positive errors are obtained. However, the errors were found to be relatively smaller than in the case of the Rühlmann and Schaffernak formulas.

In the Baticle formula (Item No. 7, Table 1) the velocity head is disregarded, and the irrational assumption is made that $A^2 R = z^5$, for a rectangular channel. The errors thus introduced are so great as to make this formula entirely unreliable.

The Poirée formula has no theoretical basis in hydraulics, but for a rectangular channel its errors were found to be rather less than those of some of the more complicated formulas.

The Tolkmitt formula applies to parabolic sections, and the effect of applying it to rectangular sections is to introduce serious negative errors.

Rough Rectangular Channels.—To simulate roughness, the flume was lined with wire netting. Other characteristics were (see Fig. 4(a)): $B = B_0 = 7.874$ in.; $S_0 = 1:500$; $D_0 = 1.97$ in.; $V_0 = 0.886$ ft per sec; $D_b = 3.937$ in.; and, in Equation (2), $m = 0.70$ and $s = 1.0$ in Equation 5(a). Therefore, $k = 0.38$; $r = 2.87$; and $K = 0.16$.

The back-water curves obtained by experiment and from various formulas are shown in Fig. 14(b), and values of $\frac{S_0\, l}{D_0}$ are shown in Table 7(b). The back-water distance in this case is somewhat greater than that for a smooth flume.

In the velocity formula as applied to rough channel sections, $m = 0.70$ actually; but in the Dupuit-Masoni formula, $m = 0.50$ so that the errors are greater than those for smooth surfaces, and values of $\frac{S_0\, l}{D_0}$ are always too large.

In the Bresse formula, errors in the ratio, $\frac{S_0\, l}{D_0}$, are rather smaller than those in the preceding formula because the errors in assuming and those introduced by assuming the rectangles to be broad cancel each other.

Values computed by the Rühlmann formula agree remarkably well with laboratory tests because the degree of roughness is great and the velocity is small; hence, the effect of neglecting the velocity head is considerably less serious than in the case of smooth channels. Moreover, errors due to the illogical nature of the formula and those introduced by assuming the rectangle to be broad, cancel each other. In this case the errors are smaller in comparison with those for the smooth channel, and the calculated values agree especially well with experimental values in the lower part of the stream where the velocity is small. However, the cancellation of various errors in this manner is rather accidental and can not be expected to occur in a general case.

Smooth, Right Triangular Channel.—The characteristics of the flume for this case were: $S_0 = 1{:}500$; $D_0 = 1.97$ in; $D_b = 3.937$ in.; $V_0 = 0.866$ ft per sec; and, $Y = 2.0$. Furthermore, in case of a triangular section, $s = 2.0$; $k = 1.0$ (and both these indices are fixed); and, $m = 0.70$. Therefore, $r = 2 \times 2 + 1.4\ (2 - 1) = 5.4$; and, $r - 2\,s = 1.4$.

Back-water curves obtained by experiment and those obtained by calculations using the writer's formula, and other formulas, are shown in Fig. 15 and values of $\frac{S_0\, l}{D_0}$ are shown in Table 7(c).

Originally, the assumption, $A^2\, R = z^5$, upon which the Baticle formula is based, was considered suitable for triangular or approximately trapezodial sections, and in spite of the fact that it neglects velocity head, this formula yields results that are fairly in accord with experiment. This illustrates the predominant importance of the shape of the section.

Positive errors due to assuming a broad section are quite serious in applying the Bresse formula to smooth, triangular channels. Since the Chezy formula is also involved, introducing other positive errors, the total is quite considerable.

The Rühlmann, Schaffernak, and Ehrenberger formulas are based on the assumption of a broad section; and the velocity head is neglected, so that

the errors are all positive and considerable. Serious errors, likewise, are introduced in working with the Poirée formula.

Rough Trapezoidal Channel.—The characteristics of the flume for this case (see Fig. 13(*c*)) were: $a_0 = 11.81$ in.; $a = \cot\theta = \cot 45° = 1$; $S_0 = 1{:}500$; $D_0 = 3.937$ in.; $D_b = 7.874$ in.; $Y = 2.0$; $Q = 0.625$ cu ft per sec; and $V_0 = 1.453$ ft per sec.

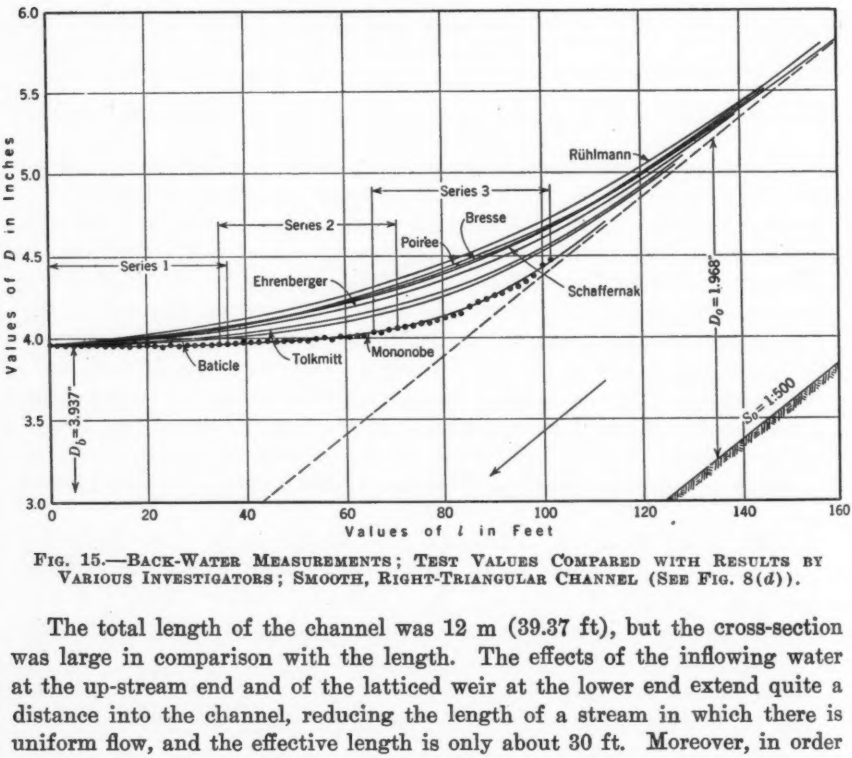


FIG. 15.—BACK-WATER MEASUREMENTS; TEST VALUES COMPARED WITH RESULTS BY VARIOUS INVESTIGATORS; SMOOTH, RIGHT-TRIANGULAR CHANNEL (SEE FIG. 8(*d*)).

The total length of the channel was 12 m (39.37 ft), but the cross-section was large in comparison with the length. The effects of the inflowing water at the up-stream end and of the latticed weir at the lower end extend quite a distance into the channel, reducing the length of a stream in which there is uniform flow, and the effective length is only about 30 ft. Moreover, in order to obtain the same ratio, $\frac{S_0\, l}{D_0}$, it was necessary that the channel be longer than a flume of less depth of water so that nine series of tests were required. Due to the skill of the operators, the results were more uniform than was expected (see Fig. 16).

In the writer's method of computation: $a = 1$; $\mu = \frac{a_0}{D_0} = 3$; $\frac{a}{\mu} = \frac{1}{3}$; and $\frac{1 + a^2}{\mu^2} = \frac{2}{9}$.

Referring to Fig. 9, the average values of s and k between the limits, $y = 1.01$ and $Y = 2.20$, are $s = 1.28$ and $k = 0.535$. In the velocity formula. m equals 0.7 so that $r = 3.60$; $r - 2s = 1.04$; and, $K = 0.286$.

Therefore, the integrals, Φ_1 and Φ_2, for any arbitrary values of $\frac{D}{D_0}$ may be read from the curves of Fig. 3. With given values of s and r introduced in Equations (21), the values of $\frac{S_0\, l}{D_0}$ (see Table 7(d)) can be determined easily, and such back-water curves as those shown in Fig. 16 can be plotted. For comparison the back-water curves based on various formulas are computed and these are shown by solid-line curves in Fig. 16.

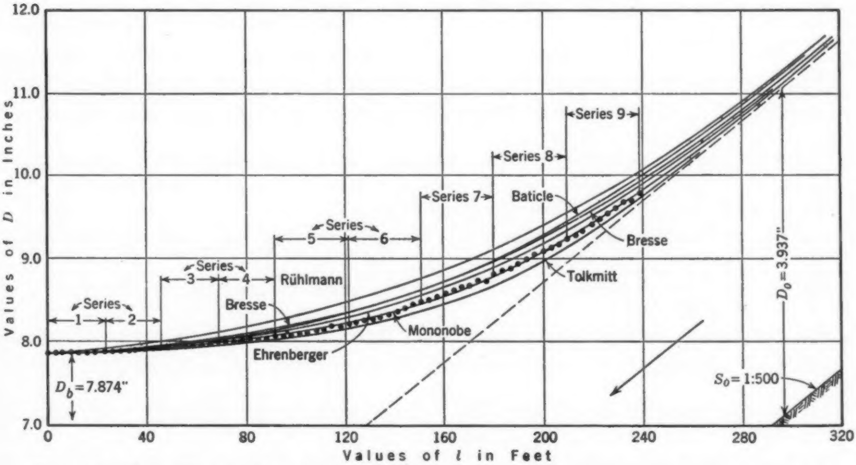


FIG. 16.—BACK-WATER MEASUREMENTS; TEST VALUES COMPARED WITH RESULTS BY VARIOUS INVESTIGATORS; ROUGH TRAPEZOIDAL CHANNEL (SEE FIG. 13(c)).

The Tolkmitt formula was derived for application to the parabolic section. Because of the analogy of this section to the cross-section of the ordinary river channel, this formula gives results that are fairly in accord with actual conditions. Positive errors are produced because $m = 0.5$ was used in the velocity formula, but errors due to the difference between sectional shapes are negative and relatively large so that the total errors are negative and are not large. In applying the Bresse formula to rough trapezoidal channels errors due to the shape of the section and the Chezy velocity formula are both positive and, therefore, fairly large.

Use of the Rühlmann formula introduces positive errors due to the shape of the section, the velocity formula, and the fact that the velocity head is neglected. The resulting errors are positive and the largest of those produced by the several formulas.

The velocity equations involved in both the Schaffernak and Ehrenberger formulas are closer to actual conditions than that in the case of the Rühlmann formula, and errors are also considerably less.

In the case of the Baticle formula, the assumption as to the shape of the cross-section is closer to actual conditions than in the case of the Rühlmann formula; but the errors are greater.

10.—Measurement of Drop-Down

In the experiments to study the drop-down, the same channels were used as in the case of the back-water except that the depth of water was more shallow and the velocity and the slope of the water surface were increased. Tests were made at depths, D_0, of 3.937 in. and 5.906 in.; and the slopes were $S_0 = 1:500$, $1:1\,000$, and $1:2\,000$.

Smooth Rectangular Channels.—Referring to Figs. 8(*b*) and 13(*a*), $B_0 = 7.874$ in.; $D_0 = 3.937$ in.; $D_s = 2.46$ in.; $S_0 = 1:1\,000$; $V_0 = 1.385$ ft per sec; $Q = 0.298$ cu ft per sec; and $Z = \frac{D_0}{D_s} = 1.60$.

The height of the water surface is measured at 50-cm (1.64-ft) intervals, but near the lowest end of the channel where the slope of the water surface is steep, readings were taken at 20-cm (7.874-in.) intervals; four observations, for different values of $\frac{D_0}{D_s}$, were plotted; and the drop-down curve from $D_s = 2.461$ in. to $D = 3.839$ in. was traced.

The critical depth, D_c (the limit between jet flow and ordinary flow) was, $D_c = \sqrt[3]{\frac{Q^2}{g\,B^2}} = 2.232$ in. Therefore, in order to prevent the occurrence of the transition from ordinary flow to jet flow so as to interfere with the drop-down, the depth, D_s, at the lowest end of the channel was selected as 2.46 in.

Computation of the Ratio, $\frac{S_0\, l}{D_0}$.—In a rectangular channel, such as Fig. 4(*a*), in which $m = 0.70$; and $s = 1.0$, the average value of k between $Z = 1.60$ and $z = 1.01$ is determined from Fig. 9, as follows: $a = 0$; $\mu = \frac{B_0}{D_0} = 2$; $\frac{1 + a^2}{\mu^2} = \frac{1}{4}$; and $k = 0.47$. Therefore, $r = 2.74$, and $K = 0.202$.

The integrals, Ψ_1 and Ψ_2, for a series of z-values are read from Figs. 5, 6, and 7, for $r = 2.74$ and $s = 1.0$; the ratio, $\frac{S_0\, l}{D_0}$, is obtained by substituting these values in Equation (20); and l is computed by simple, algebraic transposition. If the curve corresponding with r and s lies between two adjacent curves of the integrals, Ψ_1 and Ψ_2, the problem is solved by interpolation.

As is evident from Figs. 6 and 7(*a*), the values of Ψ_2, expressing the effect of the change of the velocity head, are relatively quite important for drop-down measurements as compared with back-water measurements. It is only by multiplying them by the coefficient, $K<1$, that the values of Ψ_2 are made smaller than Ψ_1 which expresses the friction loss (see Figs. 5 and 7(*b*)). Consequently, it is clear that all the current drop-down formulas in which the velocity head is neglected, are entirely inadequate.

Referring to Figs. 5 and 7(*a*), the increase in Ψ_1 as compared with the increase in z is very small when z exceeds a certain value. For instance, the difference of Ψ_1 between $z = 3.0$ and $z = 2.0$ is almost equal to zero. Accord-

ingly, the actual distance between the two sections is very small, which means that the water surface falls so considerably as to outlaw the formulas now in current use. Moreover, the application of this method of computation is limited to the part of the channel in which the flow is normal; it is not applicable to the parts of the channel shallower than D_c, such as the case in which $z \gtreqless 2$ at the lower end of the channel.

Test Measurements for the Drop-Down Curve is Smooth Rectangular Channels.—Test values for drop-down, with corresponding values derived by current formulas in use, are plotted in Fig. 17(*a*), and corresponding values of $\frac{S_0\, l}{D_0}$ are shown in Table 8(*a*).

TABLE 8.—DROP-DOWN MEASUREMENTS; COMPARISON OF THE RATIO, $\frac{S_0\, l}{D_0}$, DETERMINED BY EXPERIMENT AND BY FORMULAS

Values of $z = \frac{D_0}{D}$ (1)	Values of D, in inches (2)	Laboratory test (3)	Mononobe (4)	Tolkmitt* (5)	Bresse (6)	Kozeny (7)	Schaffernak (8)	Rühlmann (9)
(*a*) SMOOTH RECTANGULAR CHANNEL								
1.50	2.628	0.0053	0.0046	......	0.0039	0.0030	0.0113	0.0154
1.40	2.815	0.0132	0.0130	......	0.0131	0.0102	0.029	0.0390
1.30	3.030	0.0376	0.0390	......	0.0324	0.0257	0.057	0.0771
1.25	3.150	0.0628	0.0635	......	0.0494	0.0392	0.082	0.1059
1.20	3.280	0.0916	0.095	......	0.074	0.059	0.117	0.146
1.15	3.425	0.140	0.138	......	0.114	0.0924	0.165	0.204
1.10	3.580	0.201	0.196	......	0.182	0.151	0.248	0.299
1.05	3.752	0.307	0.344	......	0.323	0.275	0.423	0.488
1.01	3.898	......	0.765	......	0.714	......	......	0.969
(*b*) ROUGH RECTANGULAR CHANNEL								
1.9	2.073	0.0026	0.0033	......	0.0018	0.0013	0.0032	0.0040
1.8	2.190	0.0064	0.0069	......	0.0045	0.0036	0.0072	0.0096
1.7	2.317	0.0118	0.0122	......	0.0092	0.0073	0.0127	0.0171
1.6	2.462	0.0195	0.0205	......	0.0158	0.0128	0.0209	0.0277
1.5	2.628	0.0350	0.0346	......	0.0268	0.0214	0.0323	0.0431
1.4	2.815	0.0613	0.0578	......	0.0448	0.0354	0.0500	0.0667
1.3	3.030	0.0956	0.0952	......	0.0757	0.0602	0.0780	0.1048
1.2	3.150	0.152	0.164	......	0.134	0.107	0.138	0.174
1.1	3.280	0.292	0.321	......	0.270	0.219	0.269	0.326
1.05	3.425	0.449	0.507	......	0.439	0.370	0.440	0.516
1.01	3.580	......	1.005	......	0.896	......	......	0.997
(*c*) SMOOTH, TRIANGULAR CHANNEL								
1.25	3.156	0.0052	0.0063	0.0100	0.0214	0.0146	0.0250	0.285
1.20	3.28	0.0137	0.0132	0.0248	0.0515	0.0413	0.060	0.0686
1.15	3.424	0.0258	0.0245	0.0528	0.0996	0.0816	0.108	0.128
1.10	3.58	0.053	0.051	0.0988	0.178	0.151	0.191	0.223
1.05	3.75	0.117	0.115	0.202	0.337	0.283	0.367	0.410
1.01	3.9	0.285	0.307	0.494	0.769	......	......	0.892
(*d*) ROUGH TRAPEZOIDAL CHANNEL								
1.40	4.215	0.0030	0.0019	......	0.0054	0.0049	0.0177	0.0236
1.30	4.545	0.0121	0.0098	......	0.0224	0.0177	0.0457	0.0617
1.25	4.72	0.0269	0.0175	......	0.0368	0.0291	0.0707	0.0905
1.20	4.915	0.0458	0.0310	......	0.0584	0.0478	0.1057	0.1306
1.15	5.14	0.0713	0.0532	......	0.0953	0.0772	0.154	0.189
1.10	5.36	0.097	0.096	......	0.158	0.131	0.237	0.283
1.05	5.615	0.220	0.191	......	0.290	0.255	0.414	0.472
1.01	5.85	......	0.490	......	0.660	......	......	0.954

*** Blank spaces indicate values not determinable.**

In the Kozeny formula, the velocity head is considered and $m = 0.70$ in the velocity formula, but the section is assumed to be a broad rectangle so

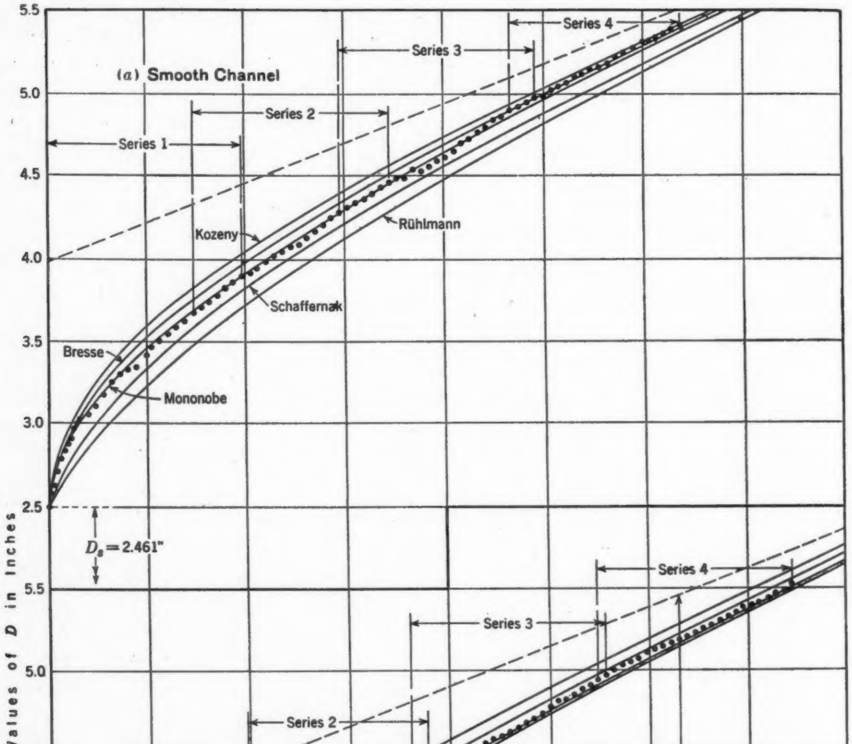


FIG. 17.—DROP-DOWN MEASUREMENTS; TEST VALUES COMPARED WITH RESULTS BY VARIOUS INVESTIGATORS; RECTANGULAR CHANNEL (SEE FIG. 4(*a*)).

that $\frac{S_0 l}{D_0}$ becomes very small. In other words, negative errors are produced and they become greater toward the low end of the flume.

In the Bresse formula, the negative errors due to assuming a broad rectangular section, and the positive errors due to assuming $m = 0.5$, cancel each other. The resulting errors are negative, but rather smaller than those introduced in the Kozeny formula.

The value of $\frac{S_0 l}{D_0}$ is increased considerably in the Rühlmann formula due to neglecting the velocity head and to assuming that $m = 0.5$. This error is slightly reduced by the assumption of a broad section, but the ultimate errors are very great, being more than 200% in the down-stream part of the flume.

In regard to the Schaffernak formula, $m = 0.75$ is too great a value for a channel such as that used in the experiment because the degree of roughness is relatively slight, and negative errors are produced for $\frac{S_0 l}{D_0}$, so that the resulting errors are considerably less than those involved in the Rühlmann formula.

Rough Rectangular Channel.—For this test, the smooth rectangular channel was made to simulate roughness by lining it with wire netting. Other characteristics were: $B_0 = 7.874$ in.; $D_0 = 3.937$ in.; $S_0 = 1:1\,000$; $Q = 0.180$ cu ft per sec; $V_0 = 0.863$ ft per sec; $D_s = 1.968$ in.; $Z = \frac{D_0}{D_s} = 2.0$; and $D_c = \sqrt[3]{\frac{Q^2}{g\,B^2}} = 0.136$ ft.

The actual depth of water at the low end of the flume was 1.67 in. which was almost equal to the calculated values of the critical depth of water. For convenience in computing the back-water curve, the section at which the depth of water, $D_s = 1.968$ in., is used as the starting point. Applying the writer's formula: $s = 1.0$; $m = 0.70$; $a = 0$; $\mu = \frac{B_0}{D_0} = 2$; $\frac{1 + a^2}{\mu^2} = \frac{1}{4}$; and the average value of k, from $Z = 2$ to $z = 1.01$, is $k = 0.46$. Finally, $r = 2.75$; and $K = 0.078$. Values of the integrals, Ψ_1 and Ψ_2, can be read directly from Figs. 5, 6, and 7. Then, similar to the case of back-water computations, it is possible to calculate the values of $\frac{S_0 l}{D_0}$ from Equation (20).

Experimental values and calculated drop-down curves obtained by using various formulas are shown in Fig. 17(*b*), and values of $\frac{S_0 l}{D_0}$ are given in Table 8(*b*).

In computing drop-down by the Rühlmann formula, the negative errors due to assuming a broad cross-section, the positive errors introduced by neg-

lecting the velocity head, and the assumption that $m = 0.5$, cancel each other, and the errors are negligible except in the low end of the flume, but the cancellation of the errors is rather accidental.

Accidentally, the algebraic sum of the positive errors introduced by the Schaffernak formula due to neglecting the velocity head, and the negative error involved in assuming that $m = 0.75$, is almost equal to the error caused by assuming that $m = 0.5$. For this reason the result agrees well with that of the Bresse formula.

Smooth Triangular Channel.—The characteristics of the flume for this test were: $S_0 = 1:1\,000$; $D_0 = 3.94$ in.; $D_s = 3.03$ in.; $Z = \frac{D_0}{D_s} = 1.30$; $V_0 = 1.083$ ft per sec; $Q = 0.1165$ cu ft per sec; and $D_c = \sqrt[5]{\frac{2\,Q^2}{g}} = 0.243$ in. Experimentally D_c was found to be 2.98 in., but for convenience the back-water is assumed to occur at the section where $D_s = 3.02$ in.

The computation by the writer's method is as follows: $s = 2.0$ (a constant); $k = 1.0$ (a constant); $m = 0.70$; $r = 5.4$; and $K = 0.247$.

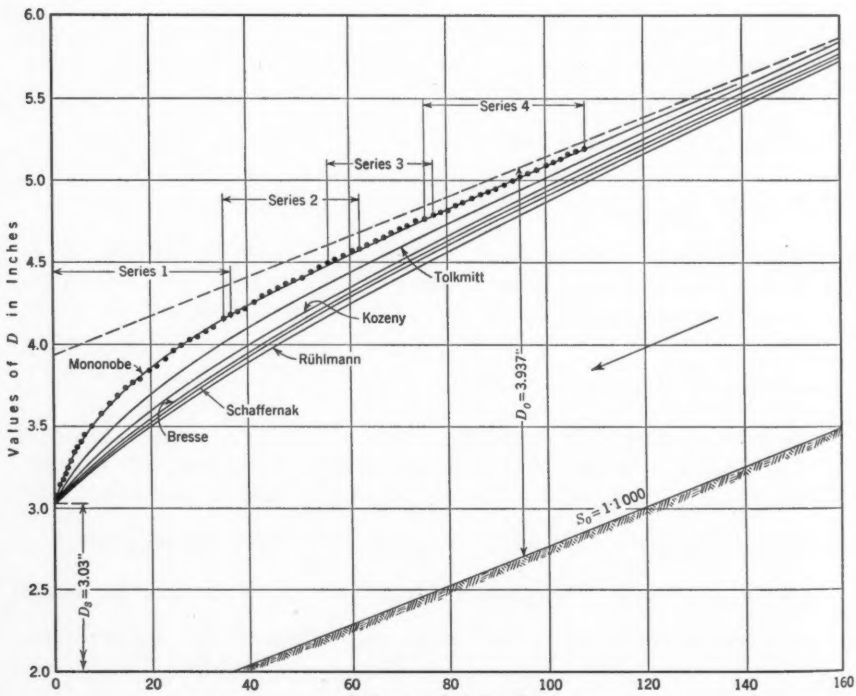


FIG. 18.—DROP-DOWN MEASUREMENTS; TEST VALUES COMPARED WITH RESULTS BY VARIOUS INVESTIGATORS; SMOOTH, RIGHT TRIANGULAR CHANNEL (SEE FIG. 8(*d*)).

As before, the integrals, Ψ_1 and Ψ_2, are read from Figs. 5, 6, and 7, and the ratios, $\frac{S_o l}{D_o}$, are computed by Equation (20). The experimental as well as the calculated drop-down curves based on various formulas are shown in Fig. 18 and the calculated values of $\frac{S_o l}{D_o}$ based on various formulas are shown in Table 8(*c*).

For the triangular channel section in general, positive errors are introduced by ignoring the velocity head, by assuming rectangular or parabolic sections, and by assuming that $m = 0.5$. Being all positive in sign these errors become quite serious.

The Kozeny formula involves the velocity head and m is assumed equal to 0.70; but the effect of assuming a broad channel section offsets the favorable conditions, and errors greater than 150 or 200% are produced. This indicates once more the importance of the shape of the cross-section in computing the drop-down.

Because of the assumption that $m = 0.5$, errors are introduced in the Bresse formula which are even greater than those in the Kozeny formula.

The Rühlmann formula applied to smooth triangular channels, involves excessive errors due to various causes, which are all positive. The result is that the total error is a serious one; for example, at the point where $\frac{D_o}{D} = 1.01$ (that is, near the extreme upper end of the drop-down), the errors are about 200% and near the lower end of the drop-down these errors are as much as 400 per cent.

In the Schaffernak formula, m is assumed equal to 0.75, so that errors are somewhat smaller than those of the Rühlmann formula.

Rough Trapezoidal Channel.—The characteristics of the flume for this case (see Fig. 13(*c*)) were: $a_o = 11.81$ in.; $a = \cot\theta = 1$; $S_o = 1:500$; $D_o = 5.91$ in.; $V_o = 1.837$ ft per sec; $Q = 1.338$ cu ft per sec; $D_s = 3.937$ in.; and $Z = \frac{D_o}{D_s} = 1.50$.

The critical depth of water, D_c, was computed as 4.016 in. and by experiment was found to be 3.937 in. By the writer's method, therefore, $a = 1$; $\mu = \frac{a_o}{D_o} = 2.0$; $\frac{a}{\mu} = \frac{1}{2}$; and $\frac{1 + a^2}{\mu^2} = \frac{1}{2}$.

Hence, referring to Fig. 9 to find the average values of s and k between $Z = 1.5$ and $z = 1.01$, $s = 1.31$; $k = 0.57$; and $m = 0.65$. Therefore, $r = 3.58$; and $K = 0.30$.

The test values for drop-down and those calculated from various formulas are shown in Fig. 19, and the calculated values of $\frac{S_o l}{D_o}$, obtained by using several formulas are found in Table 8(*d*).

In applying the Kozeny formula to rough, trapezodial flumes positive errors are introduced by assuming broad cross-sections, and small nega-

tive errors by assuming that $m = 0.70$. However, the total errors are less in this case than in the use of the other formulas, because the velocity head is taken into consideration.

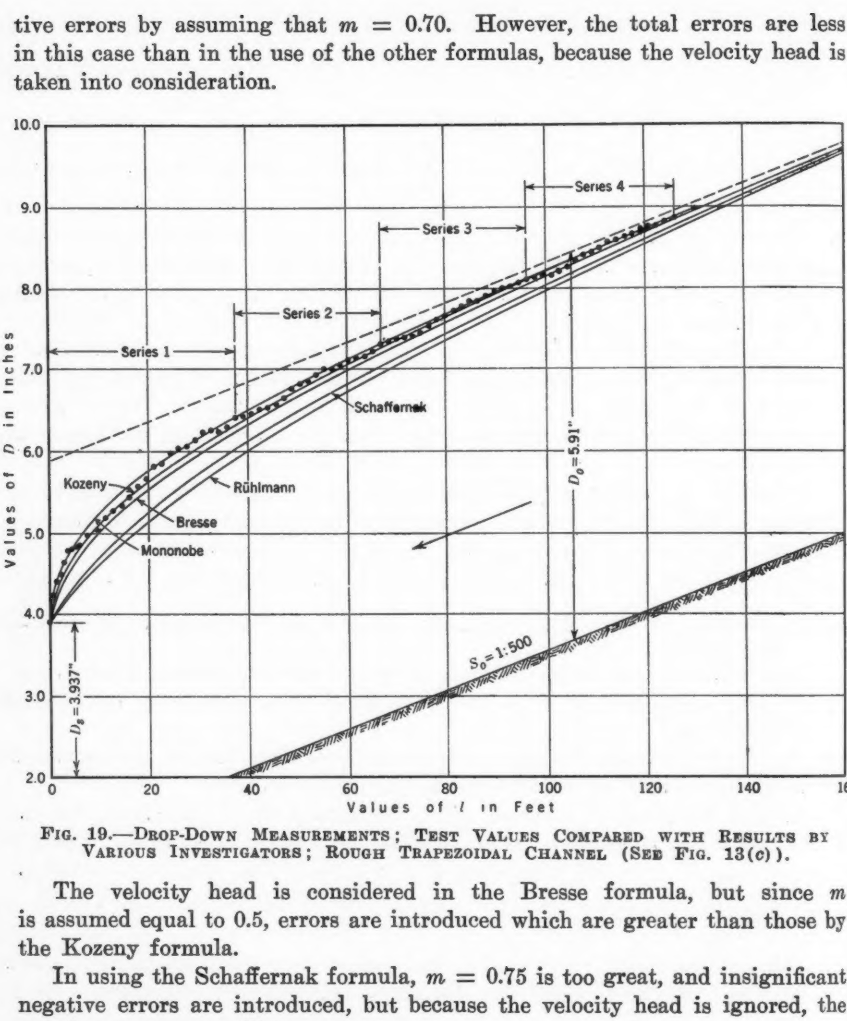


FIG. 19.—DROP-DOWN MEASUREMENTS; TEST VALUES COMPARED WITH RESULTS BY VARIOUS INVESTIGATORS; ROUGH TRAPEZOIDAL CHANNEL (SEE FIG. 13(c)).

The velocity head is considered in the Bresse formula, but since m is assumed equal to 0.5, errors are introduced which are greater than those by the Kozeny formula.

In using the Schaffernak formula, $m = 0.75$ is too great, and insignificant negative errors are introduced, but because the velocity head is ignored, the positive errors are still too great.

The total errors involved are greatest in the case of the Rühlmann formula because it introduces most of the unfavorable conditions cited.

11.—CONCLUDING REMARKS

In formulas for back-water and drop-down which are in current use the errors that enter because of illogical assumptions are:

(*1*) *Errors Introduced by Assuming C in the Chezy Formula as a Constant.*—

(*a*) In back-water computations, the actual value of C increases considerably with an increasing depth of water in a down-stream direction. Hence,

if C is assumed constant and equal to its value for the original uniform flow, the computed surface slope, S, becomes steeper than the true slope. The depth of water is constant at the weir so that the computed distance, l, between the weir and the section where the depth of water is D, becomes too small. Moreover, S_0 and D_0 are fixed values so that the ratio, $\frac{S_0 \, l}{D_0}$, is too small, and the error involved is negative.

(*b*) In drop-down computations the computed velocity is more than the actual if C for the case of uniform flow is used; consequently, the drop computed is too great and positive errors are introduced.

(*2*).—*Errors Created by Neglecting the Change in Velocity Head,* $\frac{d}{dx}\left(\frac{V^2}{2g}\right)$.—

(*a*) In the case of back-water, the velocity, V, decreases in a down-stream direction and the value of $\frac{d}{dx}\left(\frac{V^2}{2\,g}\right)$ is negative. Neglecting this factor, the computed slope, S, is steeper than the true slope corresponding to the fixed quantity of flow, Q. Since the depth of water at the lower end of the flume is constant, the calculated water level at points toward the upper end of the flume is too high; consequently, the elevation of the back-water is too high, so that $\frac{S_0 \, l}{D_0}$ becomes too small, and negative errors are produced.

(*b*) In the case of drop-down, $\frac{d}{dx}\left(\frac{V^2}{2\,g}\right)$ is positive; and if this term is disregarded, the computed slope, S, that is necessary to maintain the flow, Q, becomes small and the drop-down in the surface becomes much too great. Positive errors greater than 100% are often introduced in the values of $\frac{S_0 \, l}{D_0}$, especially near the lower end of the flume.

(*3*).—*Errors Caused by Assuming the Channel Section to Be Very Wide.*—

(*a*) In the case of back-water, the effect of assuming a wide cross-section is the same as assuming that the wetted perimeter, P, is equal to the width of the surface. That is, the hydraulic mean depth, R, is assumed to be greater than its true value. Thus, the value of S in Equation (1) becomes too small, and positive errors are introduced.

(*b*) The result of using too small a value of S, in the case of drop-down, is to make the computed water-surface slope steeper than the true surface slope and, consequently, the computed value of $\frac{S_0 \, l}{D_0}$, is too great.

The foregoing conclusions, and supporting arguments, indicate that formulas for drop-down and back-water in current use, result in ratios, $\frac{S_0 \, l}{D_0}$, which are either too great or too small. The effect of adopting illogical assumptions is to superpose positive and negative errors. In certain cases

these errors may cancel to some extent, but more often they build up into serious discrepancies of one sign or the other. For this reason many of the old formulas have yielded data involving seriously large errors, especially in the case of drop-down.

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PAPERS

DYNAMIC DISTORTIONS IN STRUCTURES SUBJECTED TO SUDDEN EARTH SHOCK

By Harry A. Williams,[1] Assoc. M. Am. Soc. C. E.

Synopsis

An inspection of earthquake records indicates that the ground may have practically harmonic motion for several successive swings after the initial shock. The amplitude and perhaps the period of the motion is then changed, sometimes quite radically. Some engineers feel that it is scarcely possible, under these conditions, for a structure with a period of vibration very close to that of the ground to attain critical distortions because of insufficient time for the amplitude to build up. This paper contains the results of model experiments on a single-mass system and a theoretical analysis of the behavior of an elevated water tank, all subjected to a sudden shock followed by harmonic vibrations. The results indicate that for single-mass structures in the near-resonance condition, critical displacements can be built up before the type of ground motion changes. There is as yet insufficient experimental evidence from which to draw similar conclusions for multi-mass structures.

Experimental Procedure[2]

The experimental investigation, which was made in the Vibration Laboratory at Stanford University, in California, dealt with the dynamic equivalent of a single-mass structure. A diagrammatic sketch of the model as installed on the shaking-table is shown in Fig. 1. The operation of the vibrating system was as follows: The table was given a sudden acceleration by dropping the pendulum against the bumper spring. When the stiff bumper spring recoiled, contact with the pendulum ceased, and the latter was withdrawn from action. The resulting motion of the table—the ground motion—was harmonic, the amplitude gradually decreasing because of friction.

Note.—Discussion on this paper will be closed in **September, 1936**, *Proceedings*.

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[2] For a complete mathematical analysis see "Vibration Research at Stanford University", by L. S. Jacobsen, *Bulletin*, Seismological Soc. of America, Vol. 19, No. 1, March, 1929.

The model consisted of a 4-in **I**-beam approximately 2 ft long, supported and guided on three balls and having a tension spring attached to each end. The model was constructed so that the amount of viscous damping (resis-

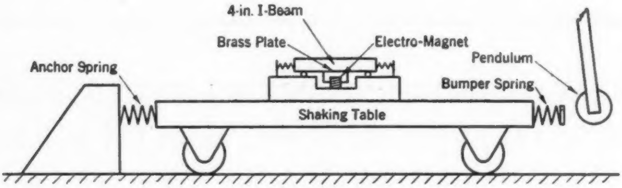

FIG. 1.

tance to vibration which is directly proportional to velocity, such as that offered by air or fluid friction) could be varied and the effect on the model motion studied, by fastening a brass plate on the under side of the **I**-beam in such a way that the plate moved through a magnetic field set up by two electro-magnets. The damping that resulted from this arrangement was directly proportional to the relative velocity of the model with respect to the shaking-table. It was found that constant or sliding friction could be neglected in the experiment.

Wires were attached to the model and to pens in such a way that records could be obtained simultaneously for the motion of the shaking-table, the absolute motion of the model with respect to the floor of the laboratory, and the relative motion of the model with respect to the table. Since the engineer is concerned only with the relative motion of a structure with respect to the ground, the records of absolute motion have not been included in this paper.

The motion of the shaking-table, which was the same for all tests, is shown by Curve 4 in Fig. 2. The pendulum was in contact with the bumper spring for 0.077 sec. During the impact the table received a maximum absolute acceleration of 14% of gravity, this maximum occurring at the end of 0.035 sec. After contact between the pendulum and bumper spring ceased, the table vibrated with a damped harmonic motion, the period of which was 0.508 sec. It reached a maximum amplitude of 0.23 in. on the first swing, at which point the acceleration was 8.5% of gravity.

All the curves of relative motion in the paper are upside down with respect to ground motion. It was necessary to take the experimental records in this form and the theoretical curves were made in the same way to be consistent.

Tests were made with three different sets of springs for the model. The spring factors were such that in the first series of tests (Fig. 2(*a*)), the free period of vibration for the model, 0.390 sec, was less than the period of the table; in the second series (Fig. 2(*b*)), it was 0.534 sec, or very nearly the same as that of the table; and in the third series (Fig. 2(*c*)), it was 0.611 sec. With each set of springs, tests were made for three different degrees of viscous damping by varying the current through the electro-magnets. Curve 1 of Fig. 2(*b*), for instance, records the test in which the viscous damping con-

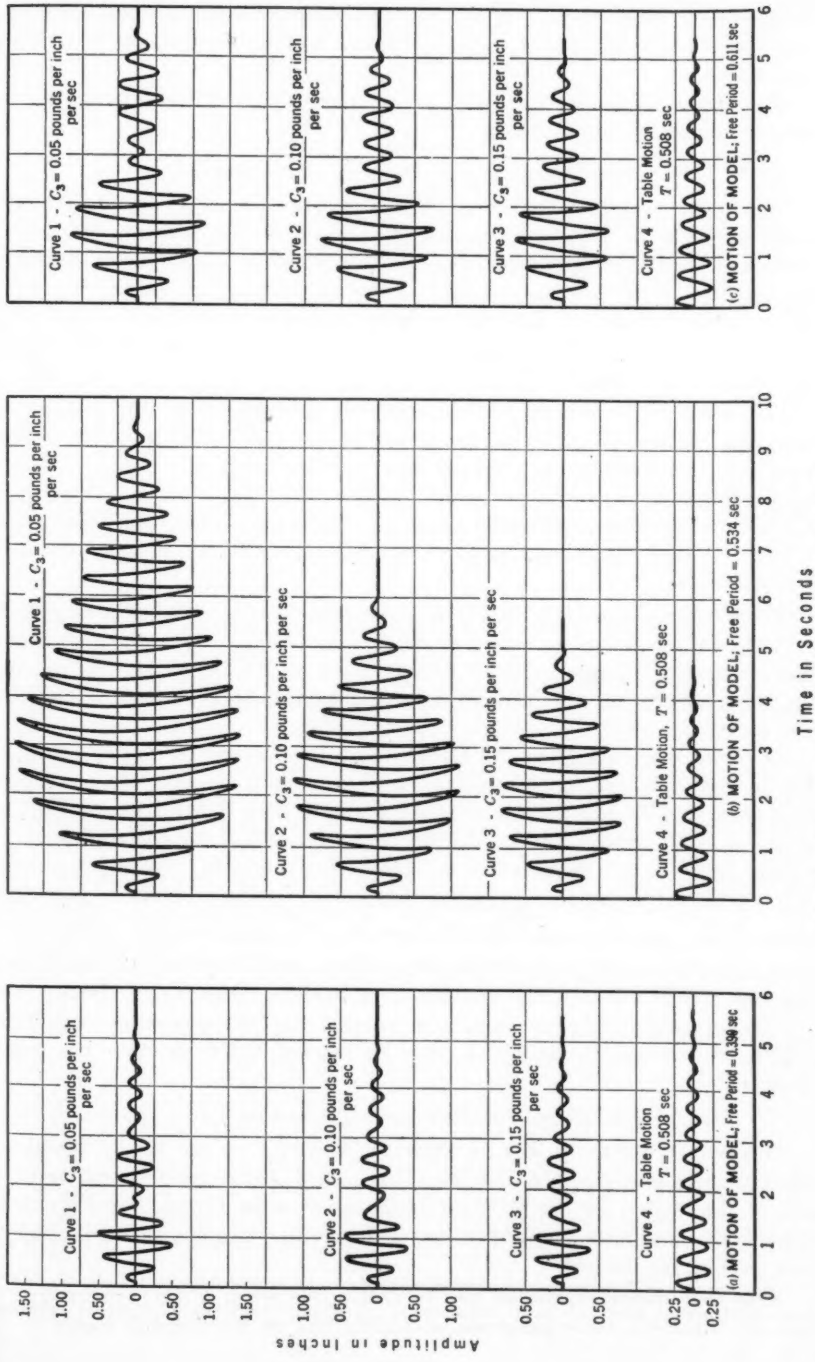

FIG. 2.—EXPERIMENTAL RECORDS OF MODEL MOTION.

stant, c_3, was quite small. It is increased from 0.05 to 0.10 and thence to 0.15 lb per in. per sec in the tests recorded in Curves 2 and 3, respectively, of Fig. 2(*b*).

Very good agreement was obtained between the experimental records and corresponding theoretical curves.

Discussion and Analysis of Test Results

When the pendulum strikes the bumper spring on the table, the latter is accelerated for a very short period—0.077 sec in the present case—after which the pendulum ceases to affect the natural free vibration of the table. The amplitudes of these free vibrations are reduced by friction until motion finally ceases. Only the first two or three complete oscillations of the table are of interest in this investigation. An examination of the records shown in Fig. 2 reveals that:

(1) When the free period of the model is somewhat less (Fig. 2 (*a*)), or somewhat greater (Fig. 2(*c*)), than that of the table, the amplitude is not large, and the maximum is attained in a very few cycles;

(2) When a near-resonance condition prevails (Fig. 2(*b*)), the amplitude is large, and the time required to reach the maximum is somewhat longer; and

(3) The effect of increasing the viscous damping factor, c_3, is to decrease the amplitude.

With reference to Item (2), it is important to notice that the amplitude built up in the first few cycles is large when compared to the maximum. The enveloping curves for these tests, plotted in Fig. 3, show that a large percentage of the maximum amplitude reached in any given test is attained very quickly.

In connection with Item (3), the effect of damping on the period of vibration is negligible. If the friction is small, the amplitude will build up somewhat more rapidly and will be greater eventually than if considerable friction is present. The curves of Fig. 3 show that friction has no appreciable effect during the first few cycles. Theoretically, this would be expected since the major part of the damping results from the presence of the exponential factor, $e^{-\frac{c}{2m}t}$, in the mathematical expression for displacement. When t is small, this factor differs but little from unity, and, therefore, the amplitude decreases slowly. Hence, the presence of a moderate amount of viscous friction in an actual structure may not prevent the attainment of dangerous amplitudes. A critical amplitude may be reached before friction has much effect.

Behavior of Model for Different Periods of Table Motion

Theoretical curves in Fig. 4 show the behavior of the model for three different table motions. In Fig. 4(*a*), the period of the table is greater than that of the model; in Fig. 4(*b*), it is the same as in Curve 3 of Fig. 2(*b*), which is the near-resonance condition; in Fig. 4(*c*), the table period is smaller than that of the model.[3]

[3] The mathematical expressions used in plotting the curves in Fig. 4 are similar to those developed by Professor L. S. Jacobsen, in his paper, "Vibration Research at Stanford University", *Bulletin*, Seismological Soc. of America, Vol. 19, No. 1, March, 1929.

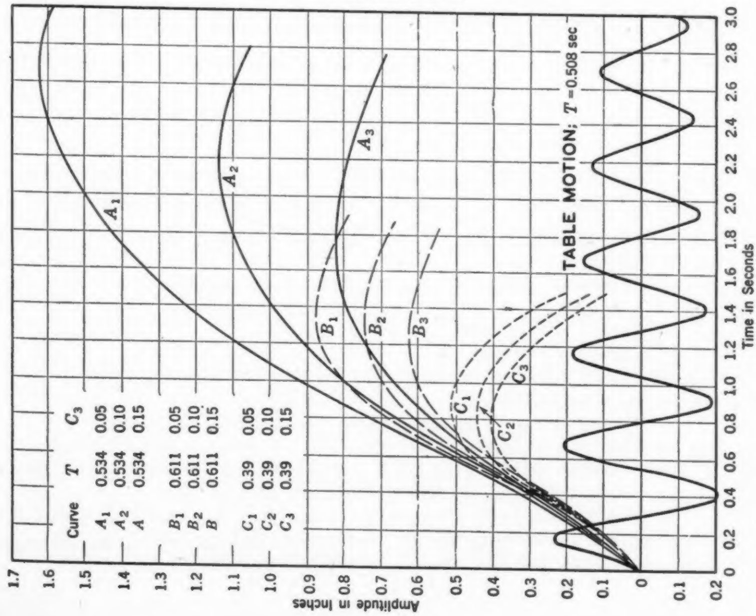

FIG. 3.—ENVELOPES OF MODEL MOTION FROM EXPERIMENTAL RECORDS.

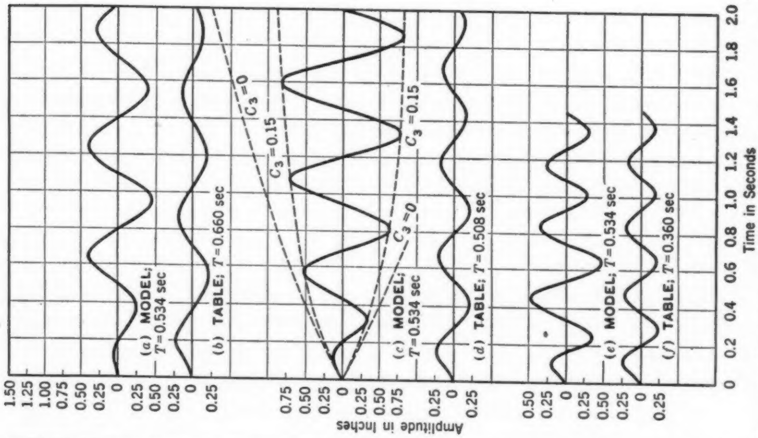

FIG. 4.—THEORETICAL BEHAVIOR OF MODEL FOR DIFFERENT PERIODS OF TABLE MOTION.

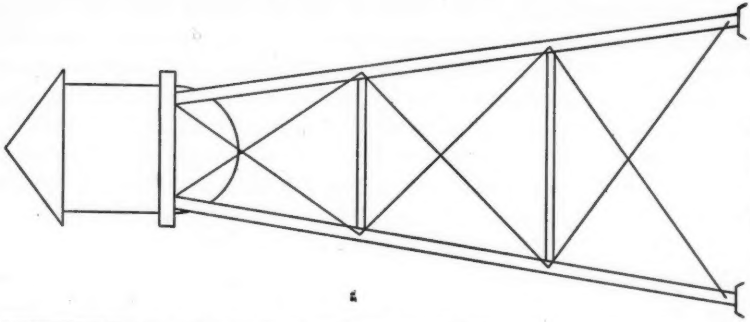

FIG. 5.—ELEVATED WATER TANK.

The result, in general, is the same as for the actual tests. As the system is removed from the near-resonance condition, the maximum amplitude is less than it is for that condition, and it is built up in a very few cycles.

It should be emphasized at this point that the distortion attained by the vibrating mass is directly proportional to the ground amplitude. Hence, it is entirely possible for an actual structure to be out of resonance with the ground and still be over-stressed because of the large amplitude of the ground motion.

Amplitude Versus Acceleration

The preceding discussion has purposely been concerned with the amplitude of the motion rather than with the acceleration. In the final analysis, the important item is the stress created in a structure by the ground movement. If the structure does not distort with respect to the ground, obviously there is no stress. A large ground acceleration may cause large distortions and, hence, high stresses; but these stresses may not actually exist until the ground acceleration has decreased almost to zero. Moreover, the force causing the high stresses may bear no relation whatever to that fictitious force so frequently used, which equals mass times acceleration.

Notation

The following symbols of this paper conform essentially with "Symbols for Mechanics, Structural Engineering, and Testing Materials", compiled by a committee of the American Standards Association, with Society representation, and approved by the Association in 1932 (the numbered subscripts are consistent with those used in the reference previously cited[2]):

c_3 = viscous damping factor for the model, in pounds per inch per second.

F = frequency, in radians per second, of the elevated water tank $= \sqrt{\frac{\kappa}{m}}$; F_2 = frequency of the table or ground motion $= \sqrt{\frac{\kappa_2}{m_2}}$; F_3 = frequency of the model $= \sqrt{\frac{\kappa_3}{m_3}}$.

G = maximum amplitude of table or ground motion, in inches.

g = acceleration due to gravity, in inches per second^2.

m = the effective mass of the elevated water tank $= \frac{w}{g}$, in pounds-second^2 per inch; m_3 = mass of the model.

r = a subscript denoting "resonance condition."

T = period, in seconds.

t = time, in seconds.

V_0 = initial absolute velocity, in inches per second, when $t = 0$.

W = total weight.

Y = displacement with respect to the ground, in inches.

y = absolute deflection, deformation of spring, or amplitude of the center of gravity of a structure, in inches; y_0 = absolute displacement of the ground, in inches.

κ = elastic constant, or spring factor for the elevated water tank; κ_2 = spring factor for the shaking-table; κ_3 = spring factor for the model, in pounds per inch.

Theoretical Behavior of an Elevated Water Tank

The model experiments show that the amplitude of motion builds up rapidly but, since the model is not intended to show stress similitude, the experiments do not indicate at what point the distortion might become critical. The behavior of a 125-ft, 50 000-gal elevated water tank is investigated mathematically with this in mind. A sketch of the structure is shown in Fig. 5. For purposes of the investigation, certain assumptions are made:

(1) It is assumed that the tank is filled with water, but only 50% of this water acts as a rigid mass in affecting the motion of the structure. This admittedly is a rough approximation. The value was selected because, at the time the investigation was begun, it seemed consistent with available data. More recent studies, together with records[4] obtained by the United States Coast and Geodetic Survey, indicate that it is probably low and that the original discrepancies between computed and actual periods were due largely to other factors. If the expressions relating to the instantaneous shock properties of water in a tank, as given by Professors Hoskins and Jacobsen[5] are applied to a rectangular tank having the same cross-sectional area and containing the same volume of water as the tank in question, a value of 78% is obtained. Professor Hoskins also computed that for shock approximately 75% of the water would act as a rigid mass in a cylindrical flat-bottom tank, 19 ft in diameter and 17.5 ft high, the same dimensions as the cylindrical portion of the water tank. The effect of a considerable error in the assumed value of 50% is discussed subsequently.

(2) In order to simplify the problem, it is assumed that the gravitational surge of water from side to side in the tank does not materially affect the behavior of the structure during the first few cycles. Tests made by the U. S. Coast and Geodetic Survey indicate that the vibrating tank tower water system may be thought of as a coupled system and that, for certain characteristic depths of water in a tank, the resulting oscillations of the system are affected considerably by the water surges even during the first few seconds.[4] Experiments made at the Massachusetts Institute of Technology with an elevated tank tower model show a "cushioning effect" which has been attributed to water surge.[6] The latter tests indicate that to neglect water surge is probably on the side of safety for the particular tank tower used for illustration. However, experimental evidence does not yet warrant the same assumption being made for elevated tanks of all sizes, or with water at any depth.

(3) The total effective mass, which includes 50% of the water, the tank, and its appurtenances, and one-half the supporting tower structure, is assumed as being concentrated at the walk line. (It is higher than this for the static condition, but if only one-half the water is considered effective during motion, it would seem reasonable to lower the center of gravity somewhat, and such an assumption does not materially affect the result in any event.)

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[4] "Observed Vibrations of Steel Water Towers", by D. S. Carder, *Bulletin*, Seismological Soc. of America, Vol. 26, No. 1, January, 1936, p. 69.

[5] "Water Pressure in a Tank Caused by a Simulated Earthquake", *Bulletin*, Seismological Soc. of America, Vol. 24, No. 1, January, 1934.

[6] "Changed Elevated Tank Design Required for Safety Against Earthquakes", by A. L. Brown, *Engineering News-Record*, October 4, 1934, p. 424.

(4) The structure is assumed to rest on a rigid foundation. (Actually, the elasticity of the ground under the footings will lengthen the period of the structure.)

(5) Viscous damping and constant friction are neglected. (Records obtained by the U. S. Coast and Geodetic Survey from the motion of elevated water tanks indicate the presence of only a small amount of friction. Damping ratios given by Mr. Carder[4] show that friction decreases the amplitude of each succeeding cycle about 1%.)

(6) Initial tension in the diagonal rods is neglected. (The effect of this assumption is discussed subsequently.)

Under the foregoing assumptions, the mass, m, of the system is 663 lb-sec² per in., and the elastic constant or spring factor, κ = 12 300 lb per in.[7] Hence, the free period of the structure is $T = 2\pi\sqrt{\frac{m}{\kappa}} = 1.45$ sec. Let it be assumed that for a number of cycles the ground has a harmonic motion in accordance with the expression:

$$y_0 = G \sin F_2\, t \qquad (1)$$

Let it also be assumed that this motion starts when the time $t = 0$; that is, the motion is assumed to start with a fictitious finite velocity from the zero displacement position. The differential equation for the absolute motion of the system then becomes:[8]

$$m \frac{d^2y}{dt^2} + \kappa\,(y - y_0) = 0 \qquad (2)$$

or, substituting for y_0:

$$m \frac{d^2y}{dt^2} + \kappa\, y = \kappa\, G \sin F_2\, t \qquad (3)$$

The solution of Equation (3) is:

$$y = \frac{\kappa\, G}{\kappa - F_2^2}\left(\sin F_2\, t - F_2 \sqrt{\frac{m}{\kappa}} \sin \sqrt{\frac{\kappa}{m}}\, t\right) \qquad (4)$$

For the resonance condition, this expression can be differentiated with respect to F_2. In the limit, $\sqrt{\frac{\kappa}{m}} = F_2$, and the result is:

$$y_r = 0.5\, G\, (\sin F_2\, t - F_2\, t \cos F_2\, t) \qquad (5)$$

However, this is the absolute motion of the assumed center of gravity. The displacement with respect to the ground is:

$$Y_r = y_r - y_0 = -0.5\, G\, (\sin F_2\, t + F_2\, t \cos F_2\, t) \qquad (6)$$

---

[7] For method of computing the elastic constant see "Computation of the Vibration Period of Steel Tank Towers", by R. S. McLean and W. W. Moore, Juniors, Am. Soc. C. E., *Bulletin*, Seismological Soc. of America, Vol. 26, No. 1, January, 1936, p. 63.

[8] See "Vibration Problems in Engineering", by S. Timoshenko, p. 13.

In Equation (6) the term, sin $F_2\,t$, will be much smaller that $F_2\,t$ cos $F_2\,t$. Hence, for all practical purposes, the motion can be expressed as:

$$Y_r = -0.5\ G\ F_2\ t \cos F_2\ t \qquad (7)$$

The resulting motion is shown by Curve *A* in Fig. 6 for the case in which the maximum amplitude of the ground displacement is 1 in. (Displacements of this order, with periods of 1 sec to 1.5 sec occurred in the Long Beach

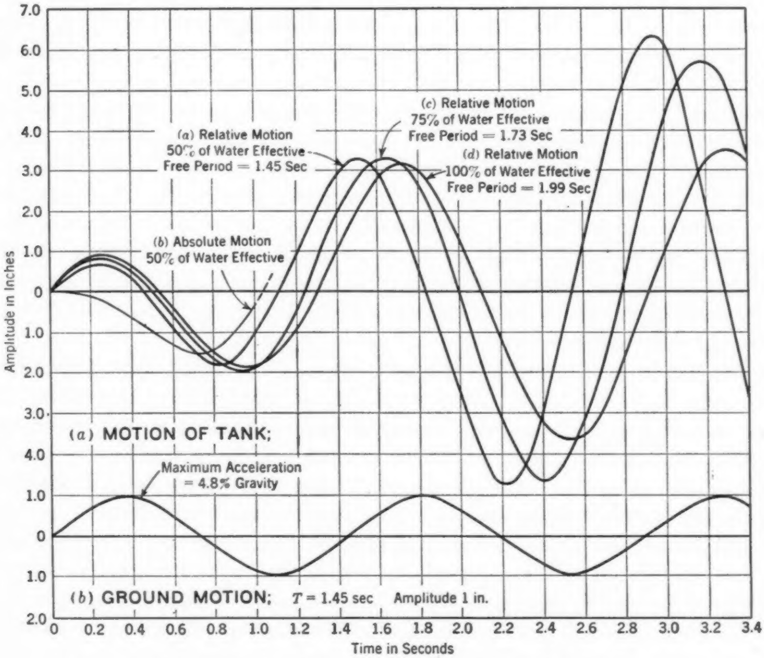

FIG. 6.—THEORETICAL BEHAVIOR OF ELEVATED TANK.

(Calif.) earthquake of March 10, 1933.) If the latter is 0.5 in., the tower displacement will be one-half that shown in Fig. 6. The maximum allowable deflection of the top of the tower, if the elastic limit in the diagonals of the top panel is not to be exceeded, is approximately 2 in. This critical value will be exceeded near the end of the first cycle of the ground motion, if the ground amplitude is 1 in., and, at the end of one and one-half cycles, if the latter is 0.5 in. The time element is too short for a moderate amount of friction to be of much help.

Admittedly, the foregoing quantitative results should be accepted with some reservations, since their accuracy is limited by the original assumptions. A study of the effect of possible errors in the assumptions, however, showed that the results are indicative of the probable behavior of the structure during the first few major swings of the ground. If 75% instead of 50% of

the water acts as a rigid mass, the free period of the structure becomes 1.73 sec. When subjected to the ground motion shown in Fig. 6, the relative motion is as shown by Curve *C*. If 100% of the water is effective, the motion is shown by Curve *D*. The expression used in plotting the latter curves was obtained by subtracting Equation (1) from Equation (4).

The behavior of the structure was studied also for the condition in which considerable initial tension was present in all diagonal rods during the first few cycles of the ground motion. The elastic constant is increased about 70% in this case and the free period is reduced to 1.12 sec. If the ground has the same period as before (1.45 sec.), the tank motion curve is of the same type as Curve *A* in Fig. 6, but the cycles are shorter and the peak amplitudes are roughly 30% less during the first two cycles.

It should be apparent that a considerable change can be made in the free period of the vibrating system and yet not eliminate the possibility of critical distortions occurring during the first few cycles, especially if large ground amplitudes prevail. As pointed out in connection with Fig. 4, the period of the ground motion can be considerably larger or smaller than the free period of a given single-mass structure with relatively little effect on the amplitude during the first cycle, or so. This statement is further substantiated by experiments made at the Massachusetts Institute of Technology with an elevated tank tower model.[6]

If the initial tension in the diagonal rods is not large, the behavior of the structure is as follows:[4, 7] As the top of the tower deflects relative to the ground, a pair of rods in the top panel drop out of action first and the elastic constant is decreased; a further deflection, and a pair of rods in the second panel drop out of action, with a consequent additional decrease in the elastic constant. On the back-swing these changes are reversed. If a rod on one side drops out of action before the corresponding rod on the opposite side, the motion is further complicated by a twisting action, which might be serious.

The behavior of the structure after the yield point has been reached in the top panel rods is even more complicated. (During the following description, the reader will do well to bear in mind a typical stress-strain diagram for structural steel.) Thus, if initial tension is negligible, the elastic constant does not vary until a critical distortion is reached and the stress in the top panel rods exceeds the elastic limit. Then this constant is decreased to a rough average value which is about 8% of its former value and the structure becomes very flexible. If the ground motion continues with the same period and amplitude, the time required for the structure to reach the end of its outward swing under these conditions depends largely on its absolute velocity at the instant the critical distortion is reached.

As the structure starts on its back-swing, the rods again increase the elastic constant for a short time until they drop out of action because of permanent elongation. That is, during the unloading, stress is again approximately proportional to strain for a steel rod which has been permanently stretched. When the center of gravity of the structure crosses the position of static equilibrium, the unimpaired rods which resist motion in the other

direction come into action. Under certain conditions, the same cycle could be repeated on that side. Once all top panel rods have been broken, the free period of the structure will be in excess of 5 sec.

There is no doubt but that the behavior of an elevated water tank beyond the critical distortion stage is an important phase of the problem. In the Long Beach earthquake of March, 1933, the top panel rods in many such tanks were badly stretched or entirely broken and yet the structures did not collapse. It is quite possible that many of the failures took place very soon after the start of the major shock and the fact that the elastic constant for such a structure no longer had a constant value but changed several times during a full swing enabled it to "weather" the remainder of the shock. Another view is that expressed by Mr. Carder.[4] He points out that, since the torsional period of most tank towers was about the same as the dominant period (0.3 sec) of the Long Beach earthquake, "it seems probable that much of the observed damage to these structures in the Long Beach region resulted from the circumstance that a large torsional motion was set up in resonance with the earth motion."

No attempt is made in this paper to plot theoretical curves for the behavior of the structure after the critical distortion has been exceeded, or to show the early behavior when there is initial tension in the diagonal rods. Any such theoretical computation should be verified by experiment.

Remarks on Behavior of Buildings

A single-story building can be expected to act as a single-mass system only if the walls are very light in comparison with the roof structure. In this case, its response to the initial shocks of an earthquake should parallel that of the model on the shaking-table. If the structure has heavy masonry walls, it usually cannot be considered a single-mass system and it is difficult to design a model that will adequately simulate its behavior.

Impact tests on a model of a multi-story building have been made by Professor Jacobsen. These tests, as yet unpublished, indicate that the distortions occurring during the first two cycles of the table motion are as large as any which occur later. Further experiment is needed, however.

Effect of Small Waves Before Major Shock

A number of waves of small amplitude usually precede the major shock. If the period of these waves is very short as compared to that of the structure the latter has very little motion, and the effect of the major shock will be practically the same as if the system had been at rest. If the preliminary waves are in resonance with the structure for a short time, the structure will be "tuned in" and the first large swing might be disastrous.

Several theoretical curves have been plotted (see Fig. 7) which show the behavior of the elevated water tank for various starting conditions. The relative amplitude for the resonance condition can be expressed as:

$$Y_r = \left(\frac{V_0}{F_2} - \frac{G}{2}\right) \sin F_2\, t + \left(Y_0 - \frac{G}{2}\right) F_2\, t \cos F_2\, t \quad \ldots\ldots\ldots (8)$$

in which Y_0 is the initial absolute displacement, in inches; and V_0 is the initial absolute velocity, in inches per second, at the instant when $t = 0$; that is, at the assumed starting point of the major ground displacements.

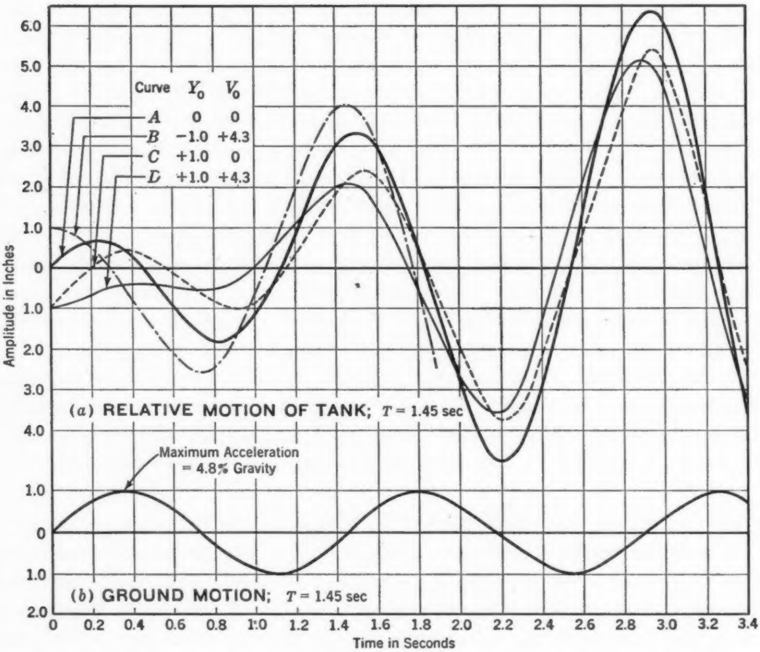

FIG. 7.—THEORETICAL BEHAVIOR OF ELEVATED TANK AT RESONANCE FOR VARIOUS STARTING CONDITIONS.

Referring to Fig. 7, Curve *A* is the same as Curve *A* in Fig. 6. The slope of this curve is not zero when $t = 0$, because of the assumption regarding ground motion. Note the slope of the curve of absolute motion in Fig. 6. It is assumed for the beginning of Curve *B* that the previous ground motion has been such that when t is assumed equal to zero, the tower has a displacement of 1 in. to the left and is moving at a velocity of 4.3 in. per sec to the right (the same direction as the ground motion on the first swing). The effect of the other starting conditions is indicated by Curves *C* and *D*.

Curves could be drawn for many other starting conditions. Some would be more and some less favorable than those shown. There is no consolation to be gained from the former because the latter are just as likely to occur. In any event, it is important to note that the rather severe starting conditions used in the example materially affected only the first one and one-half cycles. This further illustrates the fact that large motions and resonance are a dangerous combination at any time even if the condition prevails for only a few cycles.

Conclusions

The investigation indicates that if the ground is suddenly accelerated from rest or near rest, and then continues for a few cycles with a harmonic motion of constant period and amplitude, large displacements will be built up for a single-mass structure which is in the near-resonance condition. The tests made to date on a multi-story building model indicate that the same behavior can be expected of this type of structure. More study of this phase of the problem is being planned.

In the model experiments a moderate amount of viscous damping did not materially decrease the amplitudes which occurred during the first few cycles. This experimental result does not necessarily mean that friction in buildings and other structures should be neglected entirely. Friction in the latter may be proportionately greater than it was in the model.

The results of the investigation further emphasize the desirability of designing a structure with a period different from that of an expected earthquake.

Acknowledgments

The writer wishes to thank Professor Jacobsen for the many helpful suggestions and assistance given during this investigation. He also wishes to thank L. A. Elsener, Assoc. M. Am. Soc. C. E., for his suggestions and for his courtesy in furnishing detail drawings of an elevated water tank.

AMERICAN SOCIETY OF CIVIL ENGINEERS

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DISCUSSIONS

PHOTO-ELASTIC DETERMINATION OF SHRINKAGE STRESSES

Discussion

BY MESSRS. L. N. G. FILON, AND HOWARD G. SMITS

L. N. G. FILON,[16] ESQ. (by letter).[16a]—In view of the importance of the problem treated in Mr. Smits' paper,[16b], the writer has studied a xylonite model made similar to Fig. 16, with a broad strap, there being a projecting "dam" on one side only. If the loads at the ends of the strap are properly centered the longitudinal tension, Q, is sensibly uniform throughout a region where the distance from the base line, $A\,B$, of the dam (Fig. 17) exceeds one-fifth the height of the dam.

However, when the stresses are measured at the point, O, of Line $A\,B$ (Fig. 17), where the shear stress parallel to $A\,B$ vanishes, Q is found to be

FIG. 16.

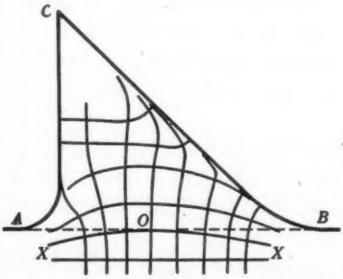

FIG. 17.—LINES OF PRINCIPAL STRESS IN DAM SUBJECT TO UNIFORM SHRINKAGE.

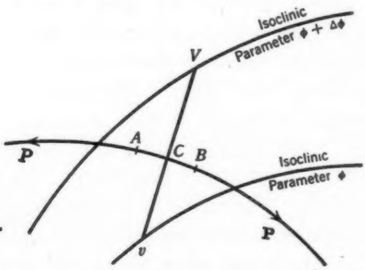

FIG. 18.—STEP-BY-STEP METHOD OF EXPLORATION OF PRINCIPAL STRESS.

(0.65) T and ($-$ 0.35) T, respectively (after applying the corrective pressure, T, below $A\,B$), on the upper and lower sides of the line, $A\,B$, of discontinuity. For the case taken by the author (T = 1 220 lb per sq in.), these forces would

NOTE.—The paper by Howard G. Smits, Jun. Am. Soc. C. E., was published in May, 1935, *Proceedings*. Discussion on this paper has appeared in *Proceedings* as follows: September, 1935, by Messrs. Thomas H. Evans, and I. K. Silverman; October, 1935, by J. H. A. Brahtz, Esq.; and March, 1936, by Arshag G. Solakian, Esq.

[16] Prof. of Applied Mathematics and Mechanics, University Coll., London, England.

[16a] Received by the Secretary February 13, 1936.

[16b] Correction to be made in Line 3 of Footnote 3 of the paper, as follows, "by Mr. Coker in a series of five articles * * *", before publication in *Transactions*.

produce a tension of 793 lb per sq in., and a pressure of − 472 lb per sq in., results which differ appreciably from those deduced from Fig. 11. Apparently, this gives 650 lb per sq in., for the tension above, and 550 lb per sq in., for the pressure below, leading to a lesser tension in the dam proper than that now found. Furthermore, the stress appears to extend deeper into the base of the dam in Mr. Smits' experiment than in the present case.

The writer has also traced the isoclinics and lines of principal stress in this case. The latter are shown in Fig. 17 and appear to be different, in the upper part of the dam, from those given by the author in Fig. 7(*b*). Fig. 7(*b*) is probably incorrect, since the lines of principal stress should approach a free boundary tangentially and normally.

The device used by Mr. Smits to solve Laplace's equation empirically is an interesting application of a well-known method originally suggested by L. Prandtl[17] in connection with the torsion of rods, and further developed by A. A. Griffith and G. I. Taylor.[18] A soap film is preferable to an india-rubber membrane, owing to the difficulty of being certain that the tension in the latter is uniform at all points and in all directions. Even if this condition is satisfied initially, it is likely to be disturbed when the india-rubber sheet is clamped between templates as described, any rapid variations of height of the edge involving appreciable pinching, which must modify the tensions postulated. This seems to be a more serious source of error than the exaggeration of the slope, as it may well not be localized.

This method assumes, of course, that the equation to be solved is Laplace's equation; in other words, that the material under consideration is perfectly elastic. The same assumption is made in applying those methods which depend on measurement of lateral extension or contraction. So far as the writer's experience goes, bakelite is a material which is very far from being perfectly elastic, and it exhibits considerable strain-creep and optical creep with time. The author does not state whether any precautions have been taken to eliminate them.

Mr. Smits is to be congratulated on having been able to obtain specimens of bakelite of adequate size apparently free from large initial double refraction. The bakelite obtainable in England is badly affected in this way, and comparatively useless for photo-elastic work.

It is not quite clear why the author (following Fig. 4) describes what he calls the method of graphical integration (but which the writer would prefer to call the step-by-step method) as "long and tedious." A similar statement appears in another paper.[19] It would seem that engineers are under a misapprehension as to the work involved in this method.

If *A* and *B* (Fig. 18) are two near points on a line of principal stress, *P*, through the middle point, *C*, of *A B*, draw a perpendicular to Line *A B*, meeting at *U*, *V*, respectively, any two neighboring isoclinics which correspond to

[17] *Physicalische Zeitschrift*, Vol. 4, 1903.

[18] *Engineering* (London), Vol. 104, 1917, pp. 658, 699.

[19] "The Stress Function and Photo-Elasticity Applied to Dams", by J. H. A Brahtz, Esq., *Proceedings*, Am. Soc. C. E., September, 1935, p. 1015.

inclinations, ϕ, $\phi + \Delta \phi$, of the polarizer and analyzer to any standard directions of reference. Then,

$$P_B - P_A = \frac{A\ B}{U\ V}\, \Delta \phi \ (Q - P)_C \ldots\ldots\ldots\ldots\ldots (11)$$

in which $\Delta \phi$ is usually a small whole number of degrees, which must be multiplied by 0.01745 to yield $\Delta \phi$, in radians.

Now, in any photo-elastic investigation, both the isoclinics and the lines of principal stress are fundamental and will have to be drawn in any case; $Q - P$ at any required point can be obtained in a variety of ways: (1) By interpolation between isochromatics; (2) by direct measurement with a compensator; or (3) by varying all loads on the model together in such a ratio that an isochromatic of a given order passes through the point. Thus, the data entering into the right-hand side of Equation (11) are immediately to hand; the value of $P_B - P_A$ is then calculated in a few moments with a slide-rule.

In this manner it is possible to work along a line of principal stress and obtain the value of P, step by step, at a number of points. If the investigator suspects inaccuracies, owing to an accumulation of errors, there is an important check, which should be applied frequently. For example, the investigator may reach the same point, using different rectangular zigzag paths along lines of principal stress, and, if the values agree (as they do when the work has been done carefully), the stresses found can be accepted with considerable confidence and used as a basis for further exploration.

There is no necessity for "graphical" integration in the ordinary sense; nor need the steps, in general, be inconveniently small. What is essential, however, is that the isoclinic lines shall have been obtained accurately in the first instance. For this, visual observation, with bright illumination and a good graticule of reference, is necessary. An ordinary photograph is usually not adequate for this purpose.

This step-by-step method[20] is entirely independent of the elastic properties of the material and is not based on the assumption that the substance used is perfectly elastic. It does, however, involve the assumption that stress difference is proportional to optical retardation, and this law generally holds well beyond the elastic limit.

HOWARD G. SMITS,[21] JUN. AM. SOC. C. E. (by letter).[21a]—The writer is gratified with the variety of viewpoints expressed by the discussers of his paper, particularly since the mathematical solution of the problem is complex enough to make the approach extremely difficult and, at best, only approximate.

Mr. Silverman has questioned the shape of the tested model, feeling that the strap was too narrow in relation to the size of the dam. Fortunately, when Professor Filon performed a similar test, he used a model similar to

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[20] Rept. of the British Assoc. for the Advancement of Science, 1923; also, *Engineering* (London), Vol. 116 (1923), pp. 511–512.

[21] Structural Designer, O. G. Bowen, Los Angeles, Calif.

[21a] Received by the Secretary March 23, 1936.

that suggested by Mr. Silverman. The difference in results obtained by Professor Filon and the writer would indicate that this geometric change in the model has bearing on the quantitative result. It is unfortunate that a relatively wider strap was not used in the original experiment.

A re-check of the material as originally presented has been made, which led to the corrections noted in the last paragraph of this discussion. The agreement between the experimental results as now correctly presented and the results obtained from the mathematical expression derived by Mr. Brahtz is very gratifying. Mr. Brahtz estimates that Equation (10) gives a value which is 10% too high due to the omission of the terms of higher order. Thus, Equation (10) may be adjusted to read:

$$\sigma_1 = 1.04\ (1 - 0.1)\ K\ E = 0.936\ K\ E \ldots\ldots\ldots\ldots\ldots (12)$$

When $K = 0.0006$ and $E = 2\,000\,000$, Equation (12) yields: $\sigma_1 = 1\,123$ lb per sq in. This is comparable to the sum of the maximum stresses on either side of the line of discontinuity at the ground line. From Fig. 11 these stresses are shown to be 525 lb per sq in. and 650 lb per sq in., giving a total of 1 175 lb per sq in. Professor Filon evaluated these two stresses at 793 lb per sq in. and 472 lb per sq in., for a total of 1 265 lb per sq in. The three values obtained (1 123, 1 175, and 1 265) fall in a narrow region, showing very good agreement between the three approaches.

Professor Filon takes exception to the lines of direction of principal stress as drawn by the writer in Fig. 7(*b*). At no time did the writer observe isoclinic lines which, when plotted, would give the sudden change of direction in the lines of principal stress, indicated in Professor Filon's Fig. 17, near the faces of the dam. It is to be noted that in all cases the lines of principal stress approach the free boundaries normally and tangentially in Fig. 7(*b*).

The advisability of using a rubber membrane is questioned by Professor Filon. Concentric circles were drawn on the rubber before stretching, and these were found to remain true circles in plan, both after stretching and after placing between the aluminum dies. It was only after observing this precaution that the writer felt justified in assuming a uniform tension in the membrane.

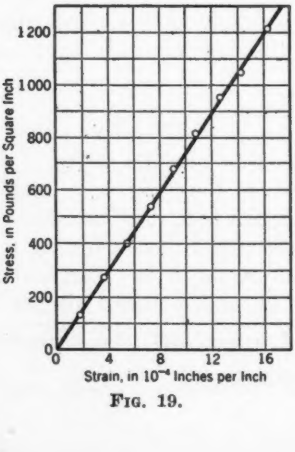


FIG. 19.

The bakelite used in this work was not free from initial double refraction until after the specimen had been carefully annealed. The annealing operation presented several difficulties. However, once properly heated and cooled, the model remained free of initial stresses for some time. In performing the experimental work on the model, the element of time was always considered to decrease the effect of optical creep.

Fig. 19 represents a stress-strain diagram for the bakelite used in the model. The relationship indicates a remarkably straight line, which showed little tendency to creep with time.

Mr. Solakian presented some very interesting material in the three conditions considered by him. It is unfortunate that more detailed information was not given, both quantitatively and qualitatively, particularly in relation to Condition (3). It is noted with particular interest that a bakelite cement has been perfected with physical qualities such that it is adaptable to photo-elastic work. The writer has long sought such a cement. The next step in approximating actual conditions with the photo-elastic method is now possible. Bakelite blocks can be cemented together simulating the manner in which they are now actually cast in the field and the entire model tested as a unit. Engineers interested in dam construction will look forward to the publication of the results of photo-elastic tests made on models built with construction joints.

The following corrections will be made when this paper is published in *Transactions*: In May, 1935, *Proceedings*, page 602, line 3 below Fig. 7, change "having already been applied" to "having not been made"; in the caption to Fig. 10, change "0.0016 inch per inch" to "0.0006 inch per inch ($E = 2\,000\,000$)"; and, in Fig. 10, of the three 400 contours, change the middle one to 600.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

FLOOD-STAGE RECORDS OF THE RIVER NILE

Discussion

BY MESSRS. KALEM OSMAN GHALEB, AND C. S. JARVIS

KALEM OSMAN GHALEB,[60] ESQ. (by letter).[60a]—Since 1926 the writer has been connected with the nilometer on Roda Island, on behalf of both the Irrigation Department and the Committee for the Preservation of Arabic Monuments.

In 1925, the column of the nilometer subsided about 7 cm and became detached from its upper support; it was propped up, to prevent it from falling. In December, 1926, pulsometers were used to unwater the well, so as to fix the column in position but, when the water level was lowered to about 2 m from the bottom of the well, dangerous cracks appeared in the walls and the unwatering had to be stopped lest the entire structure collapse. The Egyptian Parliament voted £20 000 (Egyptian) in the financial year beginning April, 1929, for the repair and restoration of the monument (see Fig. 16), as well as the expropriation of land surrounding it.

The column is an octagonal marble pillar with an enlarged base; the capital is an addition, probably of the Eighteenth Century. The engineer who erected the column in 861 A. D., has left a full description of its graduations. He states that it is divided into 19 equal cubits; in fact, at the upper end of the cubit just below the capital is inscribed in relief, in Arabic in Cufic characters, the words, "nine (&) ten cubits." As far as has been ascertained, the column is broken in three pieces: The top break, which is indicated on the drawing made during the Napoleonic Expedition, has been well repaired; the two ends of the lower break have been badly joined together, and the corresponding cubit now measures 21 cm only instead of the usual length of 54 cm. As in the case of the Suez Canal, the French savants attached to the Expedition, who have left such a wonderful detailed record of the monument, made an extraordinary mistake; in the Canal they had found a great difference in levels between the Mediterranean and the Red Seas; in the Roda nilometer they mistook the top cubit for the sixteenth; and one of them—the famous Arabic scholar, Marcel—when he returned to

NOTE.—The paper by C. S. Jarvis, M. Am. Soc. C. E., was published in August, 1935, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: November, 1935, by Halbert P. Gillette, M. Am. Soc. C. E.; December, 1935, by Messrs. R. W. Davenport, H. E. Hurst, Thomas H. Means, J. W. Beardsley, and J. C. Stevens; and January, 1936, by Jesse W. Shuman, M. Am. Soc. C. E.

[60] Insp.-Gen. of Irrig., Lower Egypt, Cairo, Egypt.

[60a] Received by the Secretary April 2, 1936.

FIG. 16.—VIEW OF RODA NILOMETER AS PROPPED UP IN 1925 TO PREVENT IT FROM FALLING.

take service in Egypt years later, indicated that mistake and read the top cubit as the seventeenth. When the well is completely unwatered it will be possible to ascertain whether the French embedded the lowest part of the column, or whether they replaced it on the floor, as formerly, and simply neglected to take account of its base in counting the number of cubits to the top.

However that may be, the reason for the apparent unpredecented rise during the final century of record of another 6 cubits, mentioned in the paragraph following Table 1 of the paper, can now be easily explained; the 3 full cubits suppressed by the French savants, become 6 cubits in the readings for the maxima stages of the river, which are above the sixteenth cubit, where half cubits are read as full cubits. The French thought accordingly that lands watered at the beginning of the Nineteenth Century, when the Nile level was at 16 cubits, were previously irrigated at between 13 and 14 cubits. Another similar error seems to be current now; it is stated in the paragraph following Table 1 of the paper, that the flood-stages that indicated an assurance of plenty progressed gradually from 16 to 20 cubits on the Roda gauge. This statement must be due to a confusion in the translation of the mediaeval chronicles, the three stages of the rise of the river corresponding with the three categories into which the agricultural land of Egypt was divided being taken for one stage only.

The low lands were watered when the Nile reached 16 cubits; the middle lands were irrigated at 18 cubits; and the high ones at 20 cubits. On the occasion of each of these stages there was a special festival, and water was let into a canal by the cutting of a cross-bank. The first of these festivals was for the "Wafa", mentioned by Mr. Jarvis, the second was for the "Neirouz" (the New Year's day of the agricultural year, which has now become the beginning of the Coptic year); and the third for the "Saleeb" which falls seventeen days after, and which is now also kept as a religious Coptic festival.

As far as historical times are concerned, the 16 cubits of the Nile have allowed a sufficient supply; that is, there was no chance of starvation when that level of the river was reached. The Arabs, on conquering the country, adopted this same criterion and called it the "Wafa." Consequently, there has been no rise in the level of the water required to irrigate the land and, therefore, that land itself cannot have risen appreciably. The writer considers that deducing the secular rise of the Nile flood-plain by dividing the thickness of a local deposit by a number of centuries results in showing Egypt to be much younger than it really is and that (as mentioned by the author under the heading "The River Nile in Egypt") other methods of reckoning, etc., almost invariably indicate much longer periods of geo-morphic development and are more accurate.

Fig. 17 shows the niches in the western wall of the well by which the gauge-reader was guided at the beginning of the rising flood; the "Wafa" stage was reached when the water attained the small bottom niche to the right, as the sixteenth cubit coincides with the bottom of the rounded cornice which touches the top of the arch of the main opening. At that stage

the column could only indicate the thirteenth or fourteenth cubit due to the misreading of the graduations; that is, considering the highest cubit as the sixteenth or the seventeenth instead of the nineteenth. This led the gauge-

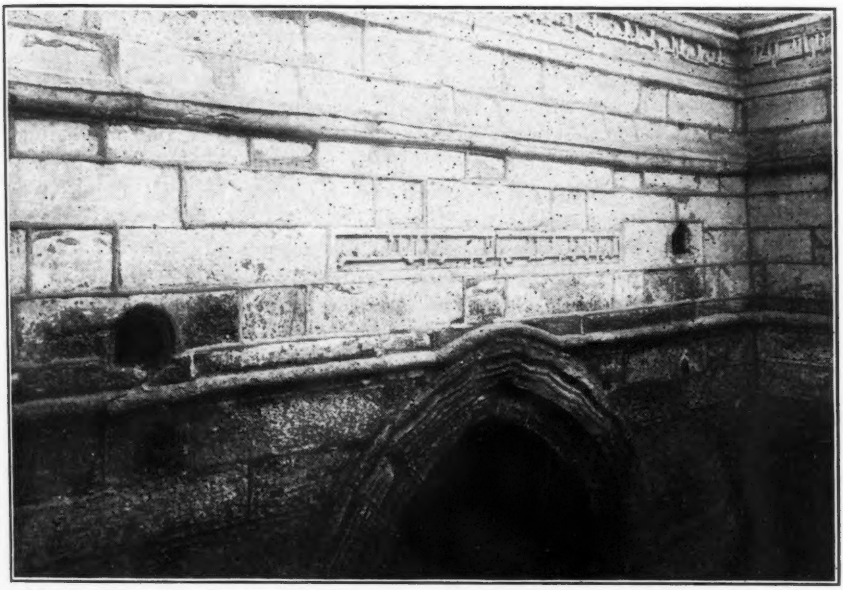


FIG. 17.—WESTERN WALL OF WELL SHOWING METHOD OF GRADUATING THE NILE GAUGE.

reader to add the difference to the "supposed" correct reading of the column, and also to consider the length of the cubits from the sixteenth to the twenty-second as one-half that of the ordinary cubits (this method of reckoning is imaginary, as there are no such graduations as those mentioned in the text following Fig. 1 of the paper; it is amusing to note how a scientific meaning can be attributed to a popular error). The seventeenth cubit coincides with the bottom of the lower Arabic inscription and the highest niche.

The Egyptian peasant may be illiterate, but he has known, from time immemorial, the level at which his land is watered; he will say, for example: "My land is irrigated at the fifteenth cubit" meaning that it will get its water when the Nile rises to that cubit. This is actually the case for the land still remaining under basin irrigation in Upper Egypt. It is the custom to design the level of the floor of a head-sluice of a flood canal to a given cubit, corresponding to the level of the land it has to water. It seems futile, therefore, to resort to the exaggeration of the Nile flood levels and create false records for increasing revenues; a "bad" government can force peasants to pay undue taxes without resorting to such a farce.

The Roda gauge records under review indicate the yearly highest and lowest readings of the level of the Nile at Cairo. The minimum readings that used to occur in May and June (it has even happened that the Roda branch of the river has been quite dry in certain years) have been affected

by the regulation on the Delta Barrage during the latter part of the Nineteenth Century and, from the beginning of the present century, the stored water at Assuan has also had its effect. Consequently, the bottom underground-water passage to the well on the eastern side is now inaccessible and it is probable that the short top tunnel (which is not indicated on the drawing in the atlas prepared by the French Expedition) was added in the middle of the Nineteenth Century to replace it. The maximum levels might have been slightly affected in certain years by the emptying of basins, but the greatest error that must be corrected is that due to the misreading of the graduations of the column; it will be fortunate if it can be proved that the pillar was considered as originally designed (that is, 19 cubits), until the end of the Eighteenth Century.

From the Napoleonic invasion onward, the records are unreliable, and it is necessary to know the period during which the readings were made on a column the top cubit of which was considered to be the sixteenth; then it is necessary to know the date at which it became the seventeenth cubit. When was the column broken anew and one of its cubits considerably shortened? When were the sixteenth to the twenty-second cubits considered to be half cubits? In 1887, a metrical scale was fixed in the Roda branch of the Nile; the readings on that scale are trustworthy, but their conversion to cubits gives false results. The two following examples are to the point.

Example (1).—Sir W. Garstin, the Under Secretary of State for the Irrigation Department, writes[61] when comparing the flood of 1892 with those of previous years: "The recorded maximum gauges of 1874 and 1878 are appallingly high, but it is more than probable that they are exaggerated, and incorrect." Referring to the year 1887, Sir Garstin continues: "Working on the above lines, it may then be assumed that the maximum height reached at Roda was 25 cubits 15 digits in 1874 (instead of 26 cubits 12 digits as recorded) and 25 cubits 14 digits, in 1878 (instead of 26 cubits 6 digits)."

Example (*2*).—When Sir William Willcocks was editing an Arabic technical review in the early Nineties, one of his party of engineers (engaged on precise leveling in connection with the fixing of the site for the Assuan Dam) contributed an article on land levels in Egypt. Among other items of information he states that, on October 15, 1893, the gauge reading of the Roda nilometer was 20 cubits 7 digits, which corresponds to a level of 18.04 m according to the method of reading indicated by Falaki (the well-known astronomer), whereas the reading on the 1887 metrical scale is 18.50 m. Where is the truth?

C. S. Jarvis,[62] M. Am. Soc. C. E. (by letter).[62a]—The generous response to this paper's presentation as indicated by the various discussions, each contributor working painstakingly along a separate thread and apparently disentangling some of the knots, has amply repaid the writer for the effort.

[61] "Note on the High Flood of 1892", p. 9, Egyptian Govt. Press, 1893.

[62] Hydr. Engr., Soil Conservation Comm., Washington, D. C.

[62a] Received by the Secretary April 18, 1936.

Mr. Gillette's views regarding cycles as affecting rainfall and related phenomena are the product of many years of study in this field, and deserve serious consideration. Mr. Davenport's studies have resulted in definite contributions to the science of hydrology, particularly as applied to the River Nile. His comparsion of the Nile with the Mississippi River and also with the Colorado River enables one to visualize the Nile floods more clearly and to account for their seeming regularity.

Both Mr. Hurst and Mr. Ghaleb are particularly equipped with information gained from actual experience, in contrast with that collected from technical research and forming the basis of this presentation. Even the differences of viewpoint and opinion between two qualified observers, such as these, are interesting and instructive. Perhaps the graduations of the Roda nilometer will be better understood by reference to Fig. 18.[63]

Mr. Means was engaged in a study of the Nile River flood records about the same time that this paper was being prepared. His personal observations in Egypt during a period of duty and travel there some years ago probably fitted him for discussing the broad phases of the Nile River discharge so clearly and authoritatively. Mr. Beardsley's personal observations, including the photograph which he furnished (Fig. 7), showing the Roda nilometer, are valuable contributions; likewise, his drawing constituting Fig. 8.

The significance of Mr. Stevens' work in smoothing out the entire record by means of both 10-yr and 50-yr averages should increase with each reference to it. Mr. Shuman has made a strikingly clear exposition of his theories regarding the applicability of various cycles to the Nile flood records, and has added some significant historical notes.

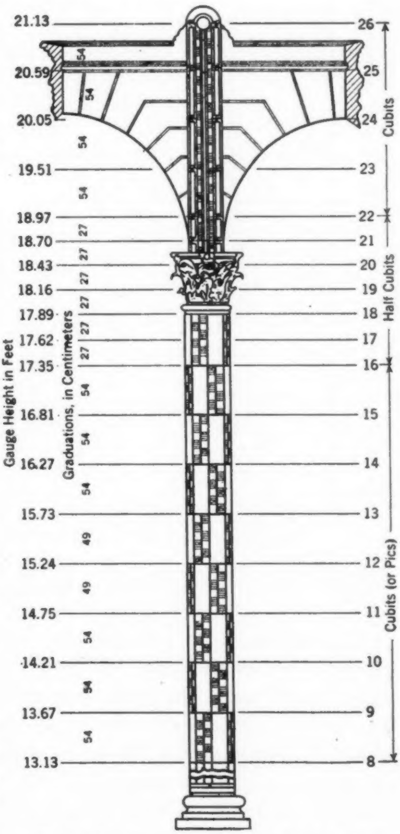


FIG. 18.—THE NILE GAUGE AT RODA.

During the preparation of the paper, the writer found the work of making complete statistical analyses of the records of the various centuries too extensive and time-consuming to complete before publication. He has since computed all the arrays, with plotting positions computed according to the modified

[63] *Memoirs*, Inst. of Egypt, Vol. 9, 1925, Pl. 21.

California method.[64] It is remarkable how nearly coincident the first seven or eight centuries of record are thereby proved to be, with gradual displacement upward to keep pace approximately with the known sedimentation rate in the Lower Valley. Fig. 19 portrays the flood-frequency trends for both the earliest and the latest century of record, and also for the 1 173 items representing the 1 300-yr period.

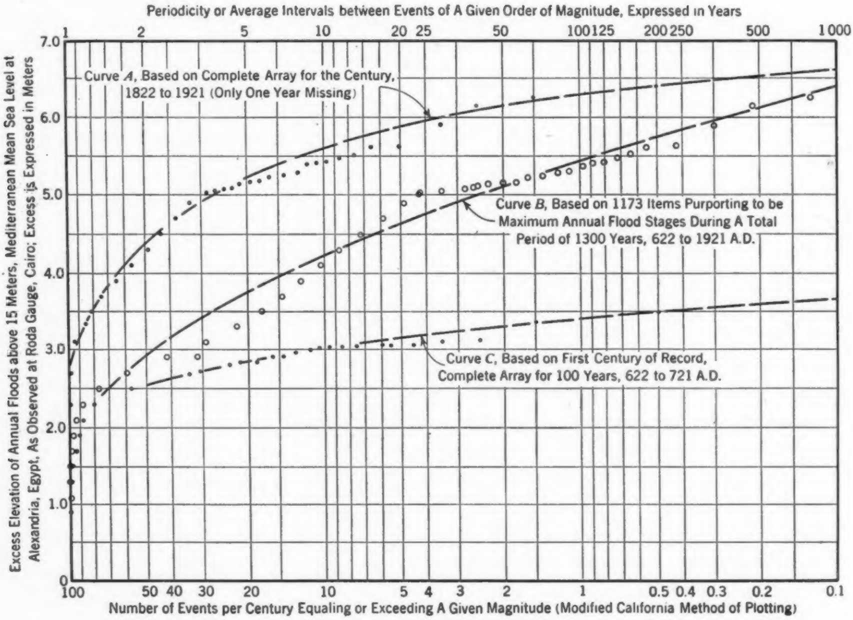


FIG. 19.—COMPARISON OF FLOOD FREQUENCY TRENDS OF THE NILE RIVER AT RODA FOR EARLIEST AND LATEST CENTURIES OF 1300 YEARS.

By reference to the statistical arrays covering the thirteen centuries of record, it is found that, of the 51 annual flood heights registering 20 m, or more, above mean sea level as observed at the Roda gauge, 30 of these occurred during the Thirteenth, or final, century, whereas 13 occurred during the Twelfth, 3 during the Eleventh, 4 during the Tenth, and 1 during the Eighth of these consecutive centuries. Of the total number of annual flood heights at or above 18.00 m, the tabulated totals for the respective centuries, listed in chronological order, are 24, 20, 22, 13, 24, 44, 33, 67, 79, 32, 37, 61, and 96. Except for the Tenth and Eleventh Centuries, all are complete, or within one or two items of complete century records. The Tenth and Eleventh Centuries have only 33 and 44 items, respectively, and should be combined for comparison with other century records. The number of annual flood events registering lower than 17.00 m for the consecutive centuries presents another series; thus 11, 20, 11, 13, 2, 4, 2, 0, 1, 1, 2, 0, and 0.

[64] *Water Supply Paper 771*, U. S. Geological Survey.

The foregoing references to the statistical arrays and plottings of the data heretofore presented in this paper and known to be affected by errors and elements opposed to comparability and homegeneity, are made for the purpose of presenting results of tedious calculations so that others may avoid repetition of the process.

Careful comparison of Roda gauge heights attained by annual floods since 1870, with flood volumes, or total annual discharge, as shown in Figs. 1 and 3 of the paper, should convince the most skeptical that Nile River flood peaks are usually fair indications of either the four-month flood discharge, or of the annual volume passing the respective gauging stations (see Figs. 20 and 21).

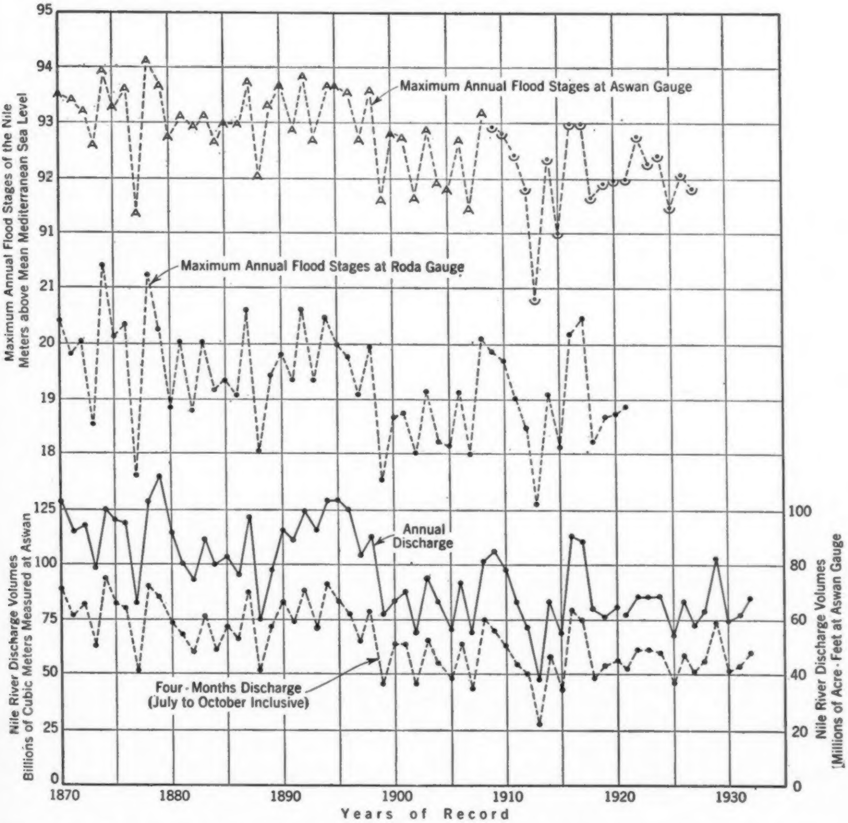


FIG. 20.—TRANSCRIPTS OF DATA FROM FIGS. 1 AND 3 FOR COMPARISON AND CORRELATION.

However, in the textual notes available in French[65], several instances are mentioned in which the peak attained "Wafa", but receded so promptly as to cause famine and distress; or, having attained a moderate stage, it continued thus in spite of the use of a portion for basin irrigation, and brought bounteous

[65] *Memoirs*, Inst. of Egypt, Vols. 4 and 9, pub. about 1925.

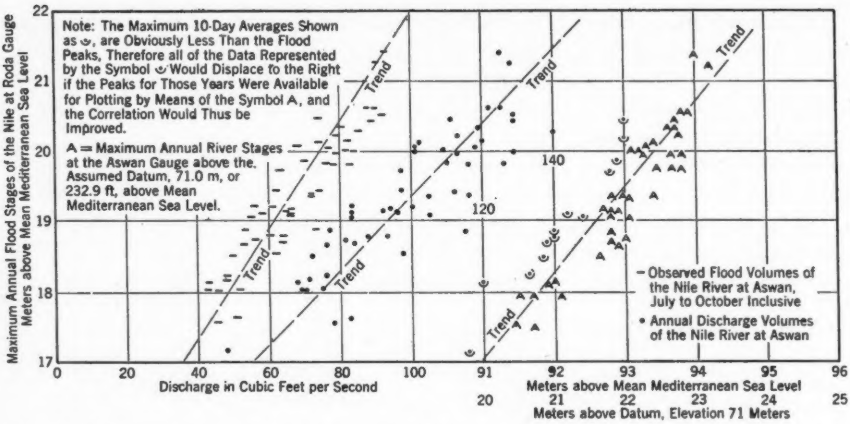


FIG. 21.—CORRELATION OF NILE RIVER DISCHARGES AT ASSUAN WITH MAXIMUM FLOOD STAGES AT RODA AND ASSUAN.

crops. Generally, however, the high-flood peak meant plentiful water supply and full harvests, unless the stage was too high; and low-flood peaks meant famine and privation. Throughout the ages of history, the gifts of the Nile have been numerous, and reasonably dependable.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

DISTRIBUTION OF STRESSES UNDER A FOUNDATION

Discussion

By Messrs. A. A. Eremin, A. Casagrande, and A. E. Cummings

A. A. Eremin,[53] Assoc. M. Am. Soc. C. E. (by letter).[53a]—Basing his computations on the Boussinesq equations, Mr. Cummings has developed some empirical formulas for computing stresses under a foundation resting on sand fill. These formulas yield stresses that are smaller than those obtained from the experiments shown in Fig. 4. The stresses at the center of the boundary area in Cases III and IV are equal to 200% of the stresses produced by uniformly distributed loading.

In Fig. 4 they are shown, erroneously, as being infinite. Evidently, stresses under a foundation computed on the assumption that the load distribution is in the form of a cone, will check experimental observations closer than those computed for Cases I to IV, inclusive.

Case V.—Conic Distribution $(n = 3)$.—In the case of a conic distribution of stress beneath a foundation, the volume of the cone equals the total load and the reaction at the center is equal to $3\,p_0$ (see Fig. 19). The equation of the cone is expressed by:

$$p = \frac{3\,p_0}{r_e}\,(r_e - r) \quad \text{..........(29)}$$

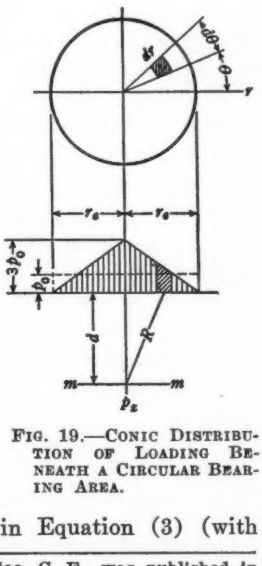


Fig. 19.—Conic Distribution of Loading Beneath a Circular Bearing Area.

Substituting Equation (29) and Equation (5) in Equation (3) (with

Note.—The paper by A. E. Cummings, Assoc. M. Am. Soc. C. E., was published in August, 1935, *Proceedings*. Discussion on this paper has appeared in *Proceedings* as follows: October, 1935, by Messrs. Clement C. Williams, D. P. Krynine, and L. C. Wilcoxen; November, 1935, by Messrs. Marshall G. Findley, and M. A. Biot; December, 1935, by Messrs. Jacob Feld, George Paaswell, and G. S. Salter; and January, 1936, by Messrs. W. S. Housel and N. M. Newmark.

[53] Assoc. Bridge Designing Engr., Bridge Dept., Div. of State Highways, Sacramento, Calif.

[53a] Received by the Secretary January 17, 1936.

$n = 3$ and $z = d$), the formula for the vertical normal stress becomes,

$$p_z = \int_{r=0}^{r=r_e} \int_{\theta=0}^{\theta=2\pi} \frac{9}{2\pi} \frac{d^3}{(d^2 + r^2)^{\frac{5}{2}}} \frac{p_0}{r_e} (r_e - r)\, r\, dr\, d\theta \qquad (30)$$

Integrating and simplifying:

$$p_z = 3\, p_0 \left(1 - \frac{d}{\sqrt{d^2 + r_e^2}}\right) \qquad (31)$$

In Equation (31), by making $d = 0$, the pressure at the boundary plane, p_z, becomes equal to $3\, p_0$, or 300% of the pressure for uniformly distributed loading.

Case VI.—Conic Distribution ($n = 6$).—Substituting Equation (29) and Equation (5) in Equation (3) (with $n = 6$ and $z = d$), the formula for the vertical normal stress is,

$$p_z = \int_{r=0}^{r=r_e} \int_{\theta=0}^{\theta=2\pi} \frac{18}{2\pi} \frac{d^6}{(d^2 + r^2)^4} \frac{p_0}{r_e} (r_e - r)\, r\, dr\, d\theta \qquad (32)$$

Intergrating and simplifying,

$$p_z = 3\, p_0 \left(1 - \frac{d^4}{4\,(d^2 + r_e^2)^2} - \frac{3\, d^2}{8\,(d^2 + r_e^2)} - \frac{3\, d}{8\, r_e} \tan \frac{-l\, r_e}{d}\right) \qquad (33)$$

By making $d = 0$ in Equation (33) the pressure at the boundary plane is $p_z = 3\, p_0$, or 300% of the pressure computed for uniformly distributed loading. At the deeper planes the pressures computed for conic distribution of loading, with $n = 6$, are close to the average stresses shown in Fig. 4, which were determined by experiments.

The Boussinesq equations have considerable theoretical value. Various simplified theories of computing the stresses under a foundation are based on them. In computing the stresses under a large foundation, however, a theory based on an assumed pyramid distribution of the stresses has considerable advantage. Paul Müller[54] has computed the stresses and deformations under a foundation in this manner and has found that such stresses and deformations are closely in agreement with those determined by the Boussinesq equations. He assumed a pyramid the sides of which were inclined at an angle of 35° to the vertical. In plastic soils the side angles vary from 35 to 90 degrees.

The problem of computing stresses under a foundation is very complicated and requires extensive experimental and theoretical study. Mr. Cummings has contributed a valuable experimental analysis.

A. CASAGRANDE,[55] ESQ. (by letter).[55a]—The question of stress distribution in a semi-infinite elastic body, which was treated so successfully by Boussinesq fifty years ago, has since been extensively elaborated. Unfortunately, most publications on this subject are little known to foundation engineers, in spite

[54] *Die Bautechnik*, 1934, p. 377.

[55] Prof., Graduate School of Eng., Harvard Univ., Cambridge, Mass.

[55a] Received by the Secretary March 31, 1936.

of the fact that these solutions could be applied to many problems in the field of earth and foundation engineering. This situation is probably due to the intricate mathematics involved. It is desirable, therefore, that the results of such theoretical findings should be presented to the Engineering Profession in a form easily understood and adapted to ready application.

The author has been successful in presenting, in a concise and clear manner, those results of Boussinesq's studies which are most important in foundation engineering. He has also presented a recent modification of Boussinesq's equations which makes possible a semi-empirical approach to the problem of stress distribution in soils that do not follow Hooke's law; and finally, he has compared the most outstanding experimental investigations with theory. This comparative analysis of experimental results and theoretical solutions is very enlightening, and its careful study is heartily recommended to those who desire information on the degree of deviation which may be expected between computed and actual stress distribution in a homogeneous mass of soil.

Mr. Cummings has confined himself to considerations of stress distribution within a semi-infinite, elastically isotropic body. Frequently the boundary conditions or the lack of homogeneity of the soil mass is such that the assumption of an isotropic, semi-infinite body represents only a rough approximation. Fortunately, mathematical solutions are available for a number of cases with more complicated boundary conditions, and even for elastically anisotropic materials, restricted only by the assumption of the validity of Hooke's law.

A condition frequently encountered is the presence of a practically incompressible stratum (rock) underlying a compressible soil stratum. Usually, it will be correct to assume the surface of the incompressible stratum to be so rough that no slippage between the two strata can occur. The other extreme would be represented by a frictionless rigid bed. In either case the vertical displacement in the elevation of the rigid surface is reduced to zero by forces acting in an upward direction, thereby increasing the concentration of stresses.

The presence of thin sand layers within an otherwise isotropic mass of clay has a restraining influence which can be idealized by the assumption of an infinitesimally thin, inextensible, flexible layer.

Following a suggestion by the writer, M. A. Biot[56] made a comparative study of the effects of such discontinuities on the stress distribution for point and line loading. For this analysis he used an original approach, although some of the solutions were found before by Michell, Melan, and Marguerre. The results are shown in Fig. 20, in which the stresses are plotted to such a scale that the maximum stress for Boussinesq's solution, is taken as unity. For the case of a vertical point load, P, the maximum normal stress on a horizontal plane at a depth, z, is expressed by:

$$\sigma = C \frac{3 P}{2 \pi z^2} \qquad \text{(34)}$$

[56] "Effect of Certain Discontinuities on the Pressure Distribution in a Loaded Soil", by M. A. Biot, *Physics*, December, 1935.

in which, in addition to the notation of the paper, σ = the maximum normal stress on a horizontal plane; and C = a constant with values as shown in Table 3. For the case of a line load, $\overline{p}$, per unit length, the maximum normal stress on a horizontal plane at depth, z, is expressed by:

$$\sigma = C \frac{2 \overline{p}}{\pi z} \qquad (35)$$

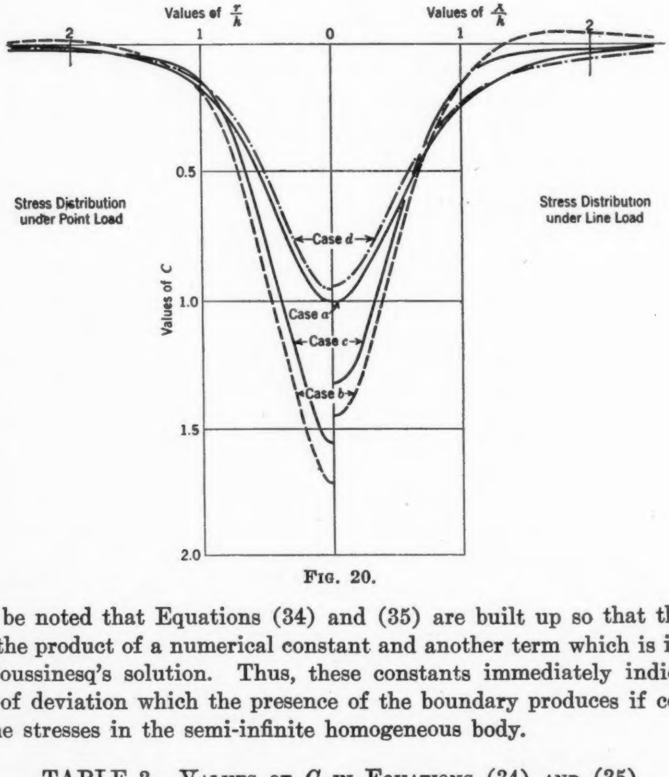


FIG. 20.

It will be noted that Equations (34) and (35) are built up so that they consist of the product of a numerical constant and another term which is identical with Boussinesq's solution. Thus, these constants immediately indicate the degree of deviation which the presence of the boundary produces if compared with the stresses in the semi-infinite homogeneous body.

TABLE 3.—VALUES OF C IN EQUATIONS (34) AND (35)

Case (see Fig. 20)	Description	Equation (34)	Equation (35)
a.	Boussinesq's distribution	1.000	1.000
b.	Frictionless rigid base	1.711	1.441
c.	Rough rigid base	1.557	1.291
d.	Inextensible flexible layer	0.942	0.935

The relative amount of concentration caused by the presence of an incompressible sub-stratum decreases with an increasing ratio of the width of the loaded area to the depth of the rigid bed. When this ratio is greater than unity, the stresses beneath the interior of the loaded area approach those computed by the integration of Boussinesq's solution.

Although the presence of a single, inextensible layer causes only a small reduction in the maximum stresses, it is likely that the presence of many such layers (as is the case, for example, in certain varved clays) would cause an appreciable spreading of the load, if compared with Boussinesq's distribution. According to Professor Biot it would be possible to solve such cases, although it would require a large number of computations. In view of the fact that such computations need to be made only once, and if available in published form, could readily be applied, they would represent a worth-while contribution to soil mechanics.

Another way to approach the problem of determining the stress distribution in stratified soils is by assuming an anisotropic material possessing a greater modulus of elasticity in the direction parallel to the planes of stratification. Solutions for anisotropic, elastic materials, corresponding to those of Boussinesq, have been found by Michel[57] and Wolf[58].

If, for example, a soil stratum consists of alternating layers of sandy silt and clay, the moduli of elasticity, E_1 and E_2, of each of these materials can be determined from tests on undisturbed samples. Assuming, for the sake of simplicity, these layers to be of equal thickness, then the average moduli of elasticity in a horizontal and a vertical direction for the entire stratum can easily be derived mathematically, and are found to be:

$$E_h = \frac{E_1 + E_2}{2} \qquad \qquad (36)$$

and,

$$E_v = \frac{2\, E_1\, E_2}{E_1 + E_2} \qquad \qquad (37)$$

Equations (36) and (37) become somewhat more complicated if the thicknesses of the alternating layers are not assumed to be equal. For example, if $E_1 = 1\,000$ kg per sq cm, and $E_2 = 100$ kg per sq cm, Equations (36) and (37) yield $E_h = 550$ kg per sq cm, and $E_v = 182$ kg per sq cm, respectively. When substituting these values in the formulas derived by Wolf[58] for a line load, one arrives at a maximum stress along the center line which is only about one-half the value for such a stress in an isotropic material. It is true that integration over a finite area and relatively shallow depth results in a smaller reduction of the stresses. However, even an average reduction of 20% or 30% in the stresses in that part of the soil which contributes most to the settlement of a structure, is of great importance in a settlement analysis, particularly when the soil is preconsolidated under loads greater than the present over-burden.

Such cases are much more difficult to evaluate analytically where a thick stratum of sand or gravel overlies a stratum of compressible soil, because the assumption of an average modulus of elasticity for the sand layer deviates materially from the actual conditions. The modulus of elasticity within the sand layer increases in proportion to the depth, and there is an additional

[57] *Proceedings*, London Math. Soc., Vol. 32, 1901.

[58] *Zeitschrift für angewandte Mathematik und Mechanik*, Vol. 15, No. 5.

increase in its magnitude beneath the loaded area, due to the stresses produced by the load. Taking all factors into account, the modulus of elasticity in the sand layer is largest below the center of the loaded area.

For such conditions, computations based on Boussinesq's stress distribution may lead to estimates of amount and distribution of settlements far in excess of those actually observed. The writer has had opportunity to study a number of such cases and he has noticed invariably that such sand and gravel layers, overlying compressible strata, reduce the magnitude of the stresses in the compressible layer by spreading the load, and are very effective in smoothing out those differential settlements which would take place if Boussinesq's equations were applicable.

An analytical approach to the problem of stress distribution in a stratum of clay that is overlaid by a stratum of granular soil is very difficult. For an approximate solution one may replace the actual thickness of the harder, upper stratum by a thicker layer of the softer underlying soil, whereby the assumed increase in thickness is a function of the ratio of the moduli of elasticity of both soils, a method suggested by Terzaghi[59].

In connection with an investigation of the probable settlements of an extension to an existing utility structure, of which accurate settlement records were available, the writer has applied the foregoing method, as well as Equation (3) in the paper, with n-values less than three, to obtain computed settlement curves which would correspond in shape to the observed settlements. These buildings are resting on a layer of sand 30 ft thick, that is underlaid by 100 ft of clay. The results obtained with both methods are practically identical. One might argue that, from a purely theoretical standpoint, the method of increasing the thickness of the upper stratum is preferable because Boussinesq's formula permits the application of the law of superposition. However, when one considers the large variations of the modulus of elasticity within the sand stratum, both methods are equally objectionable from a theoretical point of view.

The actual settlements of the edge of the loaded area are larger in relation to the settlement of the central part, than according to either of these approximate methods. Due to the larger stresses beneath the central part, the modulus of elasticity in that region of the sand is large and hence individual loads are distributed over a wider area, than the same loads near the edge, where the smaller stresses mobilize a smaller modulus of elasticity. Furthermore, the distribution of the loads near the edge is unsymmetrical with the greater concentration on the outside of the loaded area. Although it is not possible to consider these variations by either of the aforementioned approximate methods, the writer has found that one can arrive at satisfactory solutions by assuming the loads in the central part acting at a higher elevation than the loads along the edge, with a gradual transition for loads intermediately situated. This procedure is similar to the method in which the thickness of the harder stratum is increased, except that the increase is not constant, but a minimum at the edge. The writer found that, for a layer of sand over-

[59] "Bodenpressung und Bettungsziffer", von Charles Terzaghi, M. Am. Soc. C. E., *Oesterreichische Bauzeitung*, No. 25, 1932.

lying a medium stiff clay, the loads may be assumed distributed over a thickened layer, the upper border forming a semi-ellipse with the horizontal axis equal to the width of the loaded area, and the vertical axis equal to one-half the thickness of the sand stratum. This simple rule is, of course, subject to modification as more observation data become available.

The objection may be raised against this solution that the surface on which the loads are applied is not plane, and that, therefore, Boussinesq's formulas do not apply. However, it must be remembered that as soon as one deals with sand, none of the suggested methods has any scientific value, but that these methods only represent rules-of-thumb for arriving at stress distributions which are similar in character to those actually observed. There would be neither more nor less reason to assume a probability function for such a distribution, rather than Boussinesq's equation, or the modification by Griffith-Froehlich, if it can be shown that such a function also approaches the observed distributions. If one tried to retain Boussinesq's formulas, it would be merely for the sake of convenience, since very useful numerical and graphical solutions are available for such formulas.[60]

Stress computations of the foregoing type are used, in combination with the necessary data on the physical properties of the soils, for analyzing settlements and the stability of foundations.

For a settlement analysis one is often satisfied to compute the volume decrease due to the normal stresses on horizontal planes. This approach is simple but involves two approximations: First, that the rate of volume decrease is equal to the volume decrease observed in a compression test with lateral confinement (consolidation test), and is independent of the distortion of the mass; and, second, that the effect of the normal stresses on vertical planes, and of the shearing stresses on volume decrease, can be neglected.

Regarding the first assumption, in 1931, the writer[61] succeeded in demonstrating with a new type of shearing apparatus that the compressibility of clays is increased by simultaneous deformation. Unfortunately, no data are as yet available as to the amount of this increase for undisturbed clay, and for the magnitude of distortion as normally encountered in the soil underlying structures. The writer believes that for such conditions this influence is negligible.

The second assumption leads to errors which, for the majority of foundation problems, are tolerably small. Besides, in many cases, where the loading is simple, it is not necessary to introduce such an approximation, because it is

[60] Influence table by Glennon Gilboy, Assoc. M. Am. Soc. C. E., pub. in the Progress Report of the Committee on Earths and Foundations, *Proceedings*, Am. Soc. C. E., May, 1933, p. 781; tables for normal and shearing stresses, for various load conditions, by L. Jürgenson in "The Application of Theories of Elasticity and Plasticity to Foundation Problems", *Journal*, Boston Soc. of Civ. Engrs., July, 1934; and graphical solutions for distribution of normal stresses and deformations under rectangular loaded areas, by W. Steinbrenner, in "Tafeln zur Setzungsberechnung", *Die Strasse*, No. 4, 1934; and influence tables for the same case by N. M. Newmark, Jun. Am. Soc. C. E., in "Simplified Computation of Vertical Pressures in Elastic Foundations", *Bulletin*, Univ. of Illinois, Vol. XXXIII, No. 4, 1935.

[61] "Research on the Shearing Resistance of Soils", by A. Casagrande and S. G. Albert, Mass. Inst. Tech., 1932; see, also, "The Shearing Resistance of Soils", by L. Jürgenson, *Journal*, Boston Soc. of Civ. Engrs., July, 1934; "New Facts in Soil Mechanics from the Research Laboratories", by A. Casagrande, *Engineering News-Record*, September 5, 1935; and "Characteristics of Cohesionless Soils Affecting the Stability of Slopes and Earth Fills", by A. Casagrande, *Journal*, Boston Soc. of Civ. Engrs., January, 1936.

possible with equal, or even with less, work to apply directly the formulas for strain which Boussinesq and others have derived. This procedure will be illustrated by the following example which the writer has used for instruction purposes since 1932.

A circular area with the diameter, D, resting on the surface of a semi-infinite body with a constant modulus of elasticity, is loaded with p per unit of area. The theory of elasticity gives for the total displacement, Δ, of the center of the area (settlement) the following formula:

$$\Delta = \frac{p}{E} \times \frac{m^2 - 1}{m^2} \times D \qquad (38)$$

in which m represents Poisson's number.

For the extreme cases of a perfectly incompressible material, with $m = 2$, and a perfectly compressible material, with $m = \infty$, the displacements are as follows:

For $m = 2$,

$$\Delta = \frac{3\,p\,D}{4\,E} \qquad (39)$$

and for $m = \infty$,

$$\Delta = \frac{p\,D}{E} \qquad (40)$$

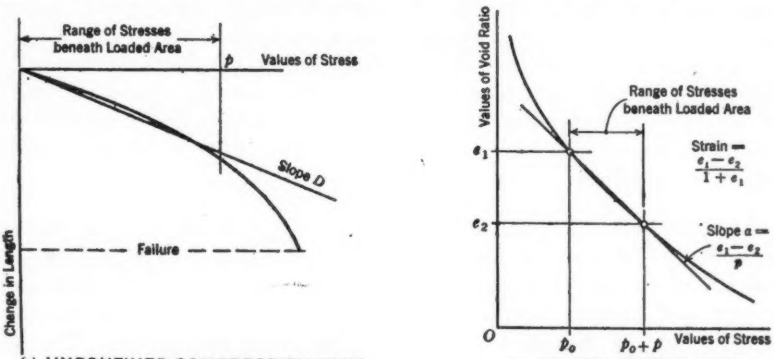
The settlement in the center, therefore, corresponds to the decrease in length of a cylinder of the same material, loaded with the same unit load, and of a height equal to 0.75; or, 1.00 times the diameter of the area.

Although in most materials, deformation and volume change cannot be separated, their relation being expressed by Poisson's number, such a separation is possible for clays. If stresses are introduced into a mass of plastic soil it will first deform without volume change, and, subsequently, a gradual volume decrease will take place at a rate which is dependent on the dimensions, drainage, and consolidation characteristics of the mass of soil.

The settlement due to immediate deformation, without volume change, can be found by utilizing the stress-strain diagram from an unconfined compression test on a undisturbed sample of the clay. A typical result of such a test is shown in Fig. 21(a). In the majority of cases the stress-strain curve for the range of the stresses which are introduced into the soil by the load, can be very well replaced by a straight line. The slope of this line is designated as the modulus of deformation, M. In its effect this modulus corresponds to the modulus of elasticity used in the theory of elasticity. With the assumption of a straight-line relationship with the same slope in the entire mass of soil one fulfills all requirements to permit the application of the theory of elasticity. Since for the deformation without volume change, $m = 2$, one arrives at the settlement in the center of the circular area, due to deformation only, by replacing, in Equation (39), the modulus of elasticity, E, by the modulus of deformation, M.

The subsequent settlement, due to gradual volume decrease, can be determined with the help of the pressure-void ratio relationship shown in Fig. 21(b),

which is obtained from a consolidation test (confined compression test) on an undisturbed sample. Since a homogeneous mass of soil is assumed, the entire mass is preconsolidated under the same load, p_0. The loading raises this



(a) UNCONFINED COMPRESSION TEST (b) CONSOLIDATION TEST

FIG. 21.

intrinsic stress in the soil in varying amounts, to a maximum of $(p_0 + p)$. For this range of stresses the change in volume may be represented with sufficient degree of accuracy by a straight line with the slope equal to the coefficient of compressibility[62], $a = \frac{e_1 - e_2}{p}$. From this value the corresponding ratio of stress over strain is derived, thus:

$$A = \frac{p}{\frac{e_1 - e_2}{1 + e_1}} = \frac{1 + e_1}{a} \qquad \text{.....................(41)}$$

which is defined as the modulus of volume change. This modulus corresponds to the modulus of elasticity of a material following Hooke's law, and with $m = \infty$. The only assumption made in this analysis is the same as that made previously, namely, that the rate of volume change is independent of deformations.

Thus, the consolidation of the mass of clay has been reduced to the behavior of a material following Hooke's law, and with $m = \infty$. Therefore, the settlement of the circular area, due to consolidation only can be found by replacing in Equation (40) the modulus of elasticity, E, by the modulus of volume change, A. Hence, the total settlement of the center of this area will be:

$$\Delta = \left(\frac{3}{2}\frac{1}{M} + \frac{2}{A}\right) p\, D \qquad \text{.....................(42)}$$

In a similar manner one can derive formulas for the settlement of other points on the surface, and for other areas.

In closing, the writer wishes to note that, in his experience, settlement due to consolidation in genuine clays can be analyzed and predicted, on the

[62] "Erdbaumechanik", von Charles Terzaghi, M. Am. Soc. C. E., Vienna, 1925.

basis of tests on undisturbed samples, with a satisfactory degree of accuracy. However, for slightly plastic silt clays, non-plastic silts, and certain organic silts, no amount of care can prevent sufficient disturbance to the samples to cause an appreciable increase in compressibility. For such materials, predicted settlements have frequently been too large.

The immediate settlement due to deformation of clay under large loaded areas was found by the writer to be always less than the values computed on the basis of unconfined compression tests on undisturbed samples. Sometimes, this discrepancy has been very large. The cause of this behavior is, at present, unexplained.

A. E. CUMMINGS,[63] ASSOC. M. AM. SOC. C. E. (by letter).[63a]—The discussion has served to bring out at least three important phases of the problem of the distribution of stresses through soils. These appear to be: (1) Questions of isotropy and Hooke's law; (2) the distribution of pressure on the contact surface; and (3) the effect of depth on the stress distribution. The writer proposes to devote this closing discussion to a consideration of these three topics.

Before considering the various topics mentioned during the discussion, however, the writer wishes to demonstrate the derivation of the stress equations for another type of surface-load distribution as shown in Fig. 22. To distinguish it from the parabolic distribution of Fig. 3(*b*), this will be referred to as an inverse parabolic distribution, varying according to the equation:

$$p = 2\, p_0 \left(\frac{r^2}{r^2_e}\right) \quad \text{.......................(43)}$$

so that the volume generated by rotating the shaded area around the vertical axis is equal to the volume of the circular disk of Fig. 3(*a*).

Case VII.—Inverse Parabolic Distribution (*n* = *3*).—Substituting Equation (43) and Equation (5) into Equation (3), with $n = 3$ and $z = d$, gives:

$$p_z = \int_{r=0}^{r=r_e} \int_{\theta=0}^{\theta=2\pi} \frac{3}{2\,\pi}\, 2\, p_0 \left(\frac{r^2}{r^2_e}\right) \frac{d^3}{(d^2 + r^2)^{\frac{5}{2}}}\, r\, dr\, d\theta \quad \text{............(44)}$$

With the method used by Professor Krynine in the derivation of Equation (25), the integration and simplification of Equation (44) leads to:

$$p_z = p_0 \left[4\, a^2 - \frac{4\, a^3}{(a^2 + 1)^{\frac{1}{2}}} - \frac{2\, a^3}{(a^2 + 1)^{\frac{3}{2}}}\right] \quad \text{............(45)}$$

Case VIII.—Inverse Parabolic Distribution (*n* = *6*).—In a manner similar to Case VII, the substitution of Equation (43) and Equation (5) into Equation (3), with $n = 6$ and $z = d$, can be shown to give:

$$p_z = p_0 \left[\frac{a^2 + 3\, a^4}{(a^2 + 1)^3}\right] \quad \text{.....................(46)}$$

[63] Dist. Mgr., Raymond Concrete Pile Co., Chicago, Ill.

[63a] Received by the Secretary April 18, 1936.

It is easily seen by substitution of $a = 0$ into Equations (45) and (46) that the vertical normal stress on the vertical center line at the ground surface is zero. This is in accordance with the load condition of Fig. 22. The manner in which this stress varies with the depth is of considerable interest and the graphs of Equations (45) and (46) are shown in Fig. 23.

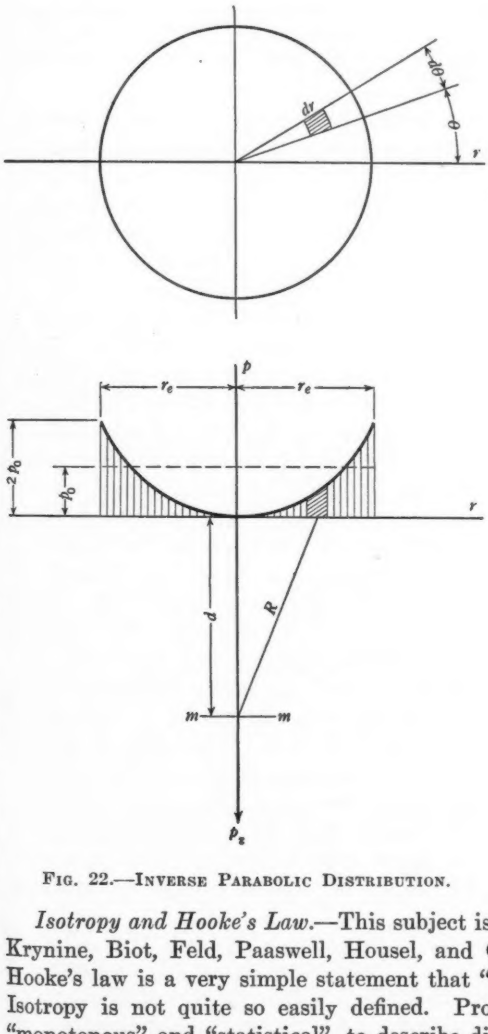


FIG. 22.—INVERSE PARABOLIC DISTRIBUTION.

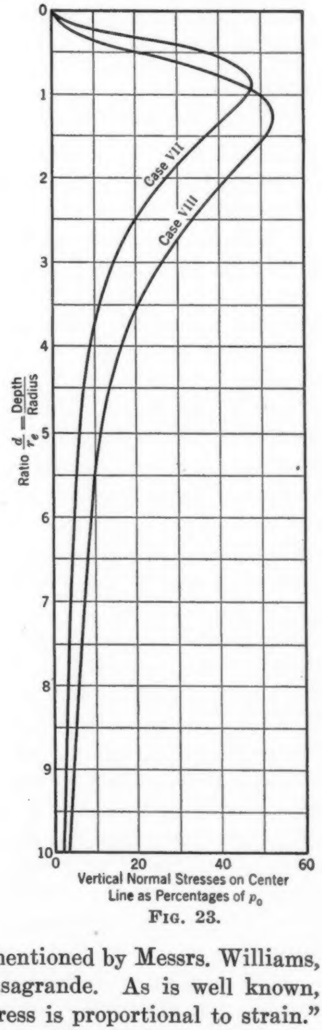


FIG. 23.

Isotropy and Hooke's Law.—This subject is mentioned by Messrs. Williams, Krynine, Biot, Feld, Paaswell, Housel, and Casagrande. As is well known, Hooke's law is a very simple statement that "stress is proportional to strain." Isotropy is not quite so easily defined. Professor Krynine uses the terms, "monotonous" and "statistical", to describe different degrees of isotropy. His conception of a "continuum" and the number of particles in a given volume connotes the idea that in some way isotropy is related to density. Mr. Feld describes an isotropic body as one "having the same physical properties in

all directions." The writer prefers Mr. Feld's statement, and believes that the chances of misunderstanding will be lessened materially by a strict adherence to the definition of isotropy commonly used in the mathematical theory of elasticity. Max Planck[64] states that,

"The necessary and sufficient condition for a body to be elastically isotropic—that is, that it should have no favored directions at all—is that its elastic constants should all be invariant with respect to any change of the coordinate system, or, what amounts to the same thing, that it can be made to coincide with itself by means of any arbitrary rotation. * * * Hence, it follows that the elastic potential of an isotropic body has only two constants which are independent of each other * * *."

It is the writer's belief that this definition precludes the idea of different "degrees of isotropy." A body is either isotropic, or it is not, and if it is not isotropic it is anisotropic or aeolotropic.

With the exception of Mr. Paaswell, all who contributed to the discussion appear to be agreed that soils do not always follow Hooke's law and that they are not isotropic in the sense in which this word is defined in the mathematical theory of elasticity. Mr. Paaswell states that the use of the Boussinesq equations "in foundation design involves the same degree of accuracy as the use of the ordinary Bernoulli formulas for flexure." These flexure formulas, of course, are based on the assumption that the displacements are infinitesimal and that the material in question follows Hooke's law. However, in the closing sentence of his discussion, Mr. Paaswell mentions the horizontal normal stresses as well as the vertical normal stress which is represented by Equation 2(c). The equations for these horizontal normal stresses include the elastic constants of the material and the sum of the three normal stresses is one of the well-known invariants of the mathematical theory of elasticity. In other words, in an elastic isotropic solid obeying Hooke's law and subjected to infinitesimal displacements which are within the elastic limits of the material, the stress system at a point within the solid includes three normal stresses and two of these stresses contain elastic constants.

For ordinary soils, Mr. Paaswell states that the terms containing these elastic constants will vanish. However, the writer is unable to understand how this problem can be expected to work both ways. According to Boussinesq, two of the three normal stress equations contain elastic constants. With the terms containing the elastic constants eliminated from the stress equations, it is evident that they are no longer those given by Boussinesq. It appears, therefore, that Mr. Paaswell agrees with the writer's Conclusion (2), namely, that the equations of the theory of elasticity must be modified before they can be applied to soils.

If it is taken for granted that soils do not obey Hooke's law and that they are not elastically isotropic, there arises the question as to what is to be done about calculating stress distributions in soils. In the writer's opinion there are several possible methods of approach: (1) By assuming that soils are anisotropic; (2) by modifying the equations for elastic isotropic solids; and (3) by eliminating the theory of elasticity in favor of a "rational method".

[64] "Introduction to Theoretical Physics", Vol. II, p. 66.

(1) The problem can be attacked on the assumption that soils are anisotropic and that it is necessary to use additional elastic constants to represent the variations of the elastic properties of the soil in different directions. This method has been demonstrated by Professor K. Wolf.[65] It is also the method used in the study of the elastic behavior of crystals.[66]

(2) The equations for elastic isotropic solids may be modified with parameters in the manner demonstrated by Griffith[9] and Froehlich.[10] The numerical values of the parameters are to be determined by comparison with experiments. This is the method used by the writer.

(3) The theory of elasticity may be eliminated entirely and the problem attacked on what may be termed a rational basis. This method is represented by the procedure outlined by Professor Housel who attributes considerable importance to surface phenomena, particularly the edge effect at the perimeter of a footing. Professor Housel's method is similar to that developed by Froehlich[67] and referred to by him as "die kritische Randbelastung." In this method the coefficient of internal friction, ϕ, plays an important part.

The interesting fact about these several methods of attacking the problem of stress distributions in soils is that, to some extent, they appear to be related in various ways. By his Equation (21), Professor Krynine shows a relationship between the Griffith-Froehlich n and Terzaghi's coefficient of pressure at rest, K. The coefficient K, in turn, is related to the coefficient of internal friction ϕ. Mr. Newmark has developed a function, ψ, which he expresses in terms of n by Equation (28). With the aid of the Rankine stress circle, Froehlich[68] has shown a relationship between the concentration factor, n, and the coefficient of internal friction, ϕ. The same writer [69] has demonstrated mathematically that the concentration factor, n, is more or less than 3, depending on whether the elastic modulus of the soil increases or decreases with the depth. By means of Fig. 16, Professor Housel compares his solution with that of Michell which is based on the elastic theory.

This problem presents an interesting field for further study particularly by means of large-scale experiments on actual structures. Information available on the subject at present is insufficient to enable one to state that any particular method of calculating stress distributions is to be preferred to all others.

Distribution of Pressure on the Contact Surface.—This problem has been discussed by Messrs. Williams, Krynine, Wilcoxen, Findley, Feld, Salter. Housel, and Eremin. There is no question but that it is an important factor in the behavior of foundations. The subject is not fully understood, and yet, of all the unsolved problems of foundation engineering, it is probably the least difficult to investigate experimentally on a large scale. It is neither difficult nor expensive to bury pressure cells under full-sized structures in

[65] "Ausbreitung der Kraft in der Halbebene und im Halbraum bei Anisotropen Material", *Zeitschrift für ange, Math. u. Mech.*, 1935, Vol. 15, No. 4, p. 249.

[66] "Mathematical Theory of Elasticity", by A. E. H. Love, Fourth Edition, p. 159.

[9] "Pressures Under Substructures", *Engineering and Contracting*, March, 1929, pp. 113–119.

[10] "Drukverdeeling in Bouwgrond", *De Ingenieur*, April 15, 1932, p. B-52.

[67] "Druckverteilung im Baugrunde", p. 83.

[68] "Drukverdeeling in Bouwgrond", *De Ingenieur*, April 15, 1932, p. B-60.

[69] "Druckverteilung im Baugrunde", p. 90.

order to determine actual pressures in the plane of contact between foundation and soil. Pressure readings continued over a period of time would show the nature of any changes that might occur in these surface-pressure distributions.

In his discussion of the problem, Mr. Wilcoxen presents Fig. 9(*b*), showing a surface distribution which is a maximum at the edges and zero in the center of the plate. Within the solid he shows a stress distribution in which the vertical stress on the vertical center line reaches a maximum of 300% of the average surface load. This maximum is shown at a depth of about one-half the diameter of the plate. Mr. Wilcoxen implies that an error has been made in the interpretation of this experiment, and that Fig. 9(*b*) represents the true explanation.

The distribution of surface pressure in Fig. 9(*b*) is the same as that of Fig. 22. The vertical normal stress on the vertical center line, for the load condition of Fig. 22, is shown in Fig. 23. It is easily seen from Fig. 23 that at no point on the vertical center line is this stress anywhere near 300% of the average surface load either in the elastic isotropic solid ($n = 3$) or in sand ($n = 6$). The distribution of surface pressure and the stress distribution within the solid, as shown in Fig. 9(*b*), are not compatible. The experiment performed by Mr. Wilcoxen is difficult to interpret, but it appears to have been an impact experiment rather than one of static pressure. The thickness of the steel test plate is not given so that it is impossible to state whether or not the plate would act as a rigid body under the blows of the fence-post.

However, some very careful experiments have been performed by Kögler and Scheidig[70] for the purpose of determining the distribution of pressure in the contact plane between a rigid body and a bed of sand. For the rigid body, they used a heavy circular block of concrete, 63 cm (24.8 in.) in diameter. Pressure cells were embedded in the lower face of the block so that the variation of pressure over the contact plane could be determined. At the same time, pressure cells were placed in the sand bed at a depth f 40 cm (15.7 in.) The block was loaded and simultaneous readings were made of the pressure distribution at the surface and of the stress distribution in the sand. The results obtained in these experiments agreed with Mr. Wilcoxen's Fig. 9(*a*). In the contact plane the pressure was a maximum at the center. Mr. Wilcoxen states that "foundation flexibility is the key to the resulting type of surface stress distribution." In the writer's opinion this is only half the problem. The other half depends on the elastic properties of the material on which the foundation is built.

A factor in this problem that is sometimes overlooked is the horizontal friction force that may be developed in the plane of contact. The point load, *P*, in Fig. 1, is normal to the surface of the solid. When Equation (3) is integrated over a finite area there is an implied assumption that the pressure under the finite area is normal to the surface of the solid and that no horizontal forces are acting in the contact plane. In the case of an actual

[70] "Druckverteilung im Baugrunde", *Die Bautechnik*, November 29, 1929, Heft 52, p. 828.

footing, it is almost certain that, under some conditions, appreciable horizontal forces are generated in the contact plane. These forces affect the stress distribution within the solid. Almost no information is available as to the nature and magnitude of these horizontal forces under a foundation or even under a test plate. However, Boussinesq[71] has discussed several theoretical solutions involving horizontal forces applied at the surface of the solid in various ways.

Messrs. Findley and Salter have discussed this question from the point of view of the practical designing engineer. Serious complications are involved, of course, in dealing with the general problem of a given foundation on a given soil. There is also the question of how far a practical designer should go in his efforts to determine the probable distribution of pressure on the contact plane. At present, the practical designer is not able to go very far in this direction due to the lack of field data as to the actual pressure distributions that exist under full-sized structures. Without these data with which to check the theoretical analysis, the problem is still more or less in the field of speculation as far as actual foundations and actual soils are concerned.

In this connection it is to be noted that Boussinesq's methods do not lead to a solution of this general problem. The surface load distributions shown in Figs. 3(*a*), 3(*b*), 19, and 22, are not to be confused with actual plates or footings. They are simply normal loads distributed over a part of the surface of the solid. If these surface distributions are known, Boussinesq's methods of applying potential functions can be used to determine the distribution of stresses within the solid. If, instead of these surface pressures, the displacements at the surface are given, Boussinesq's analysis can also be applied, and the stress distribution within the solid can be found from the surface displacements.

However, in the general problem, neither the pressure distribution nor the surface displacements is given. A footing with certain elastic properties is to be placed on a soil with other elastic properties. The footing is to be loaded by means of a concentrated column load, or otherwise. The available methods for attacking this problem and the solution of a practical problem of this type, were outlined in 1935 by Froehlich.[72] The solutions for a number of cases involving circular plates carrying various types of loads have been published by Dr. Ing. Ferdinand Schleicher.[73] The equations for stresses and displacements include the elastic properties of the plate as well as the elastic modulus of the foundation material.

The Effect of Depth on Stress Distribution.—This subject was mentioned several times during the discussion. Professor Krynine states that, "probably at a certain depth within the earth all matter obeys the Boussinesq law * * *." Mr. Newmark calls attention to the fact that "* * * the experimental data are limited to measurements at depths of less than 5 ft * * *", and that, "* * * at great depths, sand should act more nearly in the manner of an elastic and homogeneous material * * *".

[71] "Application des Potentiels", p. 72.

[72] "Die Bemessung von Flachgründungen aus Eisenbeton und die neuere Baugrundforschung", *Beton und Eisen*, 1935, Heft 12,

[73] "Kreisplatten auf elastischer Unterlage", Berlin, 1926.

It is true, of course, that the physical properties of soils depend to some extent on the pressures under which they exist. In sand, it is generally agreed that the elastic modulus increases directly with the depth. For a homogeneous sand bed of great depth, the elastic properties at a depth of 50 ft would certainly differ from those at a depth of 5 ft. It is a question, therefore, as to how far the experiments may be extrapolated for application to full-sized structures. As has been shown, the experiments indicated a value of about 6 for the concentration factor, n. Whether or not a stress concentration factor as high as 6 would apply to a full-sized structure on a deep bed of sand is largely a matter of speculation. It seems probable that, in a bed of compact sand of indefinite depth, the average stress concentration factor might not reach a value of 6, although it would be more than 3.

In discussing this question of depth, Mr. Paaswell mentions Boussinesq's theory of "local perturbations" and states that, "the manner of loading a foundation and the soil pressures near its loading can generally be ignored in the determination of the stress distribution in the deep strata * * *". This is correct if the strata under consideration are deep enough. However, when an analogy is drawn between the foundation problem and the girder problem, as is done by Mr. Paaswell, there is one important factor in the foundation problem that must not be overlooked.

"Depth" in the foundation problem is a relative term and the vertical co-ordinates of Figs. 4 and 23 are not absolute depths; they are ratios of depth to radius of loaded area. The concentrated load in the girder problem is not exactly analogous to the distributed loads in the foundation problem. In general, the stress distribution calculations are made for structures that cover ground areas on the order of 100 or 200 ft in diameter. Even if the structure were supported by isolated footings, it is the entire loaded area that would have to be considered. At the same time, the depth from ground level to rock or hardpan, over the greater part of the surface of the earth, is also on the order of 100 or 200 ft. There are well-known exceptions, of course, such as the City of Mexico and the mouths of the Mississippi and the Yangtse Rivers. However, the most common condition is one in which the depth of soil and the diameter of the loaded area are of the same order of magnitude. In other words, in a practical foundation problem, it is the region represented by the upper parts of Figs. 4 and 23 that is of particular interest. In this region the distribution of surface pressure is especially important and, regardless of the probable value of the concentration factor, there are large variations in the magnitude of the soil stresses due entirely to the manner in which the ground surface is loaded. The foundation problem, therefore, appears to be largely a question of surface phenomena and local perturbations rather than one of stress distributions in a semi-infinite solid at depths so great that the surface load conditions can be ignored.

This problem of stress distributions is also influenced by other important factors, such as those mentioned by Professor Casagrande. Rigid underlying strata; planes of discontinuity due to stratification in the soil mass; soft clay beds under beds of dense sand—all affect the stress distributions. These problems have been attacked theoretically and for some of them a theoretical

solution is available. Proof of the accuracy of the theoretical analysis must await the collection of data in the field. It is interesting to note that in the case of a rigid underlying stratum, Dr. Biot's equations (Fig. 20) indicate stress concentrations in excess of those given by the Boussinesq equation. In most of the experiments shown in Table 1 and Fig. 4, the pressure cells were placed on the more or less rigid concrete floor of the laboratory. The experiments produced stress concentrations similar to those required by Dr. Biot's analysis.

Summary and Conclusions.—The number and variety of the discussions on this paper are indicative of the great interest that has developed in foundation problems and soil mechanics in recent years. Nevertheless, there can be no question as to the truth of Professor Williams' remark concerning the "rather amorphous state of foundation literature" at the present time. However, Professor Williams sees no reason for discouragement in this situation; nor can the writer. It is believed that a parallel can be drawn between the development of soil mechanics and the development of the theory of elasticity which is the basis of modern structural analysis.

Galileo, in 1638, was the first mathematician to endeavor to determine the nature of the resistance of a beam to rupture. During the next 200 yr the development of the elastic theory was in the hands of such men as Galileo, Hooke, Mariotte, Young, Euler, Daniel Bernoulli, James Bernoulli, La Grange, Coulomb, and others. These are all well-known names and yet, in the "Historical Introduction" to his "Mathematical Theory of Elasticity," Love writes, as follows:

"At the end of the year 1820 the fruit of all the ingenuity expended on elastic problems might be summed up as * * * an inadequate theory of flexure, an erroneous theory of torsion, an unproved theory of the vibrations of bars and plates, and the definition of Young's modulus."

However, Love goes on to state that "* * * such an estimate would give a very wrong impression of the value of the older researches". The year 1821 marks the discovery by Navier of the general differential equations of elastic equilibrium. Since then another hundred years have elapsed during which elastic theory and structural analysis have made great progress in many directions.

In comparison with this record, soil mechanics and foundation engineering are very young sciences. Rankine's work is about eighty years old and his theories are now termed "classical". Boussinesq's great works on pulverulent masses and on the theory of stress distributions are only fifty years old. Terzaghi published his theory of the consolidation of clay about fifteen years ago. In the past ten years much progress has been made, although many important problems remain unsolved. The most serious obstacle in the way of further progress at the present time is the lack of accurate information as to the behavior of existing structures.

In conclusion, the writer wishes to express his sincere thanks to those who contributed to the discussion. He feels that the value of his paper was materially increased by reason of the discussion, and he believes that the discussion has demonstrated the validity of his original conclusions.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

ADAPTATION OF VENTURI FLUMES TO FLOW MEASUREMENTS IN CONDUITS

Discussion

By Harold K. Palmer, M. Am. Soc. C. E., and Fred D. Bowlus, Assoc. M. Am. Soc. C. E.

Harold K. Palmer,[25] M. Am. Soc. C. E. and Fred D. Bowlus,[26] Assoc. M. Am. Soc. C. E. (by letter).[26a]—From the many constructive discussions received, the writers are encouraged to hope that, more and more, water will be measured by Venturi flumes developing critical depths. Especially will this be true when more accurate verification can be obtained, in hydraulic laboratories, of some of the principles involved, and more simplified calculations of flow can be made, as suggested by some of the discussers.

Mr. Hopkins has developed excellent formulas for calculating co-ordinates on the quantity-energy head curves. Since these curves form the basis of all Venturi flume rating curves, the writers suggest that all notations referring to the frictional loss, varying with the shape of the throat, and to velocity head, varying with the velocity, be omitted from the terms of the formulas offered by Mr. Hopkins. This would mean substituting ε for Z in his formulas. In order that these may be readily available, the formulas for the more common throat sections are summarized as follows: For a flume of rectangular cross-section, Equation (26) becomes:

$$Q = 3.09 \; b \; \varepsilon^{1.5} \qquad (50)$$

for a flume of V-shaped cross-section, Equation (28) becomes:

$$Q = 2.297 \; p \; \varepsilon^{2.5} \qquad (51)$$

and, for a flume of trapezoidal cross-section, Equation (31) becomes:

$$Q = 3.06 \; (b + 0.72 \; p \; \varepsilon) \; \varepsilon^{1.5} \qquad (52)$$

Note.—The paper by Harold K. Palmer, M. Am. Soc. C. E., and Fred D. Bowlus, Assoc. M. Am. Soc. C. E., was published in September, 1935, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: November, 1935, by Messrs. N. F. Hopkins, and Hunter Rouse; February, 1936, by Messrs. F. V. A. E. Engel, and J. C. Stevens; and April, 1936, by Prof. Ing. Filippo Arredi.

[25] Chf. Draftsman, Los Angeles County Sanitation Dists., Los Angeles, Calif.

[26] Res. Engr., Los Angeles County Sanitation Dist. No. 2, Whittier, Calif.

[26a] Received by the Secretary April 22, 1936.

Professor Arredi has suggested a most welcome graphical method of determining the proper size for a Venturi flume throat in any given installation and for drawing the rating curve, thus obviating the more laborious method advanced by the writers.

Mr. Stevens describes a slab, curved in longitudinal section (which he has already used in circular conduits), based on somewhat the same principle as that of the writers. This would probably add some refinement by reducing frictional losses which, however, might be smaller than the recording device could register. The writers believe that the insertion of a level section in the bottom slab will be positive assurance that critical depth is obtained with parallel flow, as proved by their early experiments. They do not agree with Mr. Stevens as to the preference of the simple flat slab over the trapezoidal form of throat. The Los Angeles County Sanitation Districts had a 6-in. slab with a 3-ft level section (6 ft including transitions), installed in a 54-in. sewer, but found it necessary to change to a trapezoidal flume for greater accuracy. One difficulty with a slab is that a thickness required to give good results during peak flows may cause deposits of silt or sludge at low flows. For very small flows in a sewer, a flume with side contractions and no bottom slab would be better.

The writers also disagree with Mr. Stevens' statement that whenever the depth of tail-water below the throat of a Venturi flume is more than 65% of that above the throat, correct calculations of the flow can be obtained only by recording depths at the two points. The writers have recently been able to investigate this point by installing a flume in an irrigation ditch of the Arroyo Ditch and Water Company (see Fig. 24), and believe that within the range of accuracy of a recorder the flow can be calculated from a single head-water depth above the flume when the tail-water depth is approximately 90% of the head-water depth. Dr. Engel has found this percentage to be as high as 92. A simple analysis of losses and depths of flow through a rectangular Venturi flume installed in a conduit with a grade so flat as to cause a low velocity, tends to substantiate the correctness of the latter figure (92%). In this case the depth of water in the throat at the point of critical velocity is practically two-thirds the head-water depth, or the velocity head equal to one-third the head-water depth (neglecting the small velocity head above the throat). As the water passes from the restricted throat section into the normal cross-section of the conduit below the flume, a condition obtains similar to that in a pipe enlargement, in which case it is the usual practice to consider that the loss is 25% of the difference in velocity heads. Neglecting the slight velocity head of the tail-water, only 25% of the velocity head in the throat would be lost, or one-twelfth the total energy head. This amounts to 8% and thus would leave the tail-water depth at about 92% of the head-water depth.

The writers have observed also that if the jump, or undulations, occur in the transition section below the throat, there is no effect on the water depth above the meter, but when the undulations occur in the throat and are backed up to a point near the section of critical depth, the head-water depth

may be affected. It is best, therefore, to design the throat so that the jump will occur in the lower transition, or with the tail-water depth less than 90% of the head-water depth.

FIG. 24.—VENTURI FLUME IN WHICH JUMP OCCURS BELOW THROAT WITH NO APPARENT BACK-WATER EFFECT. VELOCITY ABOVE THROAT IS ABOUT 2.5 FEET PER SECOND, RESULTING IN SURFACE WAVES.

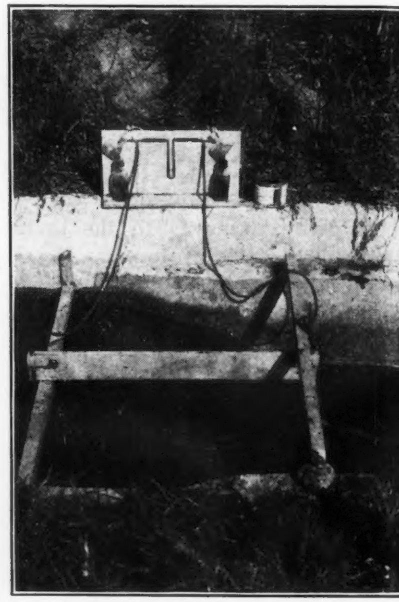


FIG. 25.—DIFFERENTIAL HEAD METER SET TO REGISTER LOSS OF HEAD AT ENTRANCE TO FLUME. IN THIS CASE LOSS WAS 0.01 FOOT FOR 10 CUBIC FEET PER SECOND.

The writers are indebted to Mr. Stevens for the details of his float mechanism which should give accurate readings without being affected by floating débris.

Dr. Engel claims, in his English experiments, an accuracy for Venturi flumes of 99.5%, but unfortunately in the two cases for which he published the data, the water did not flow at critical depths in the throat and, therefore, the results are not applicable to the present case. The accuracy attained seems to vary inversely as some power of the velocity above the throat. Dr. Engel questions the accuracy of the Parshall meter, but in the United States it is generally accepted as a standard method of measuring water, and the device has been legalized by the Division of Water Rights of the California Department of Public Works. The writers have apparently been misunderstood in the test comparison with the Parshall meter. As the wooden Venturi flume was installed in an irregular channel the comparative results with the Parshall meter were only submitted to show the general adaptability of the flume throughout a considerable range in flow, in a sewer where it would have been impossible to have installed a weir. Dr. Engel states that the important point regarding losses due to the installation of a Venturi flume

in a conduit has been entirely neglected by the writers. In the writers' method of computing a rating curve, the loss of energy up stream from the throat affects the rating curve, whereas the loss down stream has no effect as long as it does not cause a backing up of the water. In the formula for the Venturi flume as given by Dr. Engel, the loss of head up stream is taken care of in the constant, C_f, but it is probably a function of the length of the throat and flow, being negligible for smaller throats and also for small flows.

Mr. Rouse has ably discussed several of the theoretical limitations to the proposed method of using these flumes, which apply in cases where circumstances permit more than the ordinary accuracy of measuring the depth of water, or in very large installations. The writers had some of these limitations in mind when they suggested measuring the loss of energy at the entrance to the throat. He rightly questions its suitability as a control meter in hydraulic laboratory experiments as it would be necessary to calibrate each installation of that kind. Under such cases the weir would serve just as well and would be much easier to install, but in the field where allowable grade loss and funds for an elaborate metering station are limited, and especially where the need for measuring the flow arises after the completion of a conduit, the Venturi flume appears superior.

His most serious objection is that the exact location of the critical depth section is unknown, and varies with the flow; this is true, but it is the minimum energy rather than the critical depth that the writers use; and small variation from critical depth, or some curvature of surface, may occur without an appreciable effect on the energy-head depth. There is a difference in the height of the energy head at the control section in the throat and the point of measurement, depending on the hydraulic conditions in this section of the conduit; but it is small and can be measured easily and allowed for within the degree of accuracy of the recording device (see Fig. 25). Although, theoretically, it is a drawback, practically the error is less than the sensitivity of the recording devices now available.

The following case illustrates some of the difficulties met in field installations. In a 24-in. trapezoidal flume set in a 54-in. sewer having a grade of 0.15%, a flow averaging 21.5 cu ft per sec produced a record with a blurred line nearly 0.25 in. in width, due to small surface waves superimposed on longer waves with a period of a minute or more. The flow varied from 20.6 to 22.4 cu ft per sec and the mean could be obtained only by estimating the middle of the trace.

Mr. Rouse takes exception to the writers' attempt at measuring flow over a great range by the method proposed. This difficulty is one which must be met in sewage-flow measurements where the daily maximum may be three times the minimum, and during a storm a much greater range is to be expected. If the sewerage system is growing, the daily maximum ten years hence may easily be ten times the minimum of to-day. In storm drains the flow is even more erratic. This precludes the use of the hook-gauge for measuring depths and substitutes the stage recorder; therefore, in considering the degree of accuracy to be attained, it must be borne in mind that refinements too small to be shown on the record need not be considered.

Mr. Rouse feels that this investigation should have been performed in a laboratory, with which the writers fully agree, but no laboratory was available, and there was great need of finding some method of installing meters in constructed sewers where weirs had been found inadequate. After the investigation had developed a practical meter, it was discussed with other engineers in the Southwest who showed great interest in it and urged that discussions and results be published at once. The writers are confident that flumes carefully constructed in accordance with their recommendations will give accurate results.

However, it is hoped that laboratory work will be undertaken to investigate some of the following uncertain points: (*a*) Proper length of throat; (*b*) formula for energy loss in head-water; (*c*) maximum ratio of tail-water depth to head-water depth; (*d*) proper angle of transition to axis of conduit; and, (*e*) accuracy of measurement as a function of velocity of approach.

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DISCUSSIONS

THE STRESS FUNCTION AND PHOTO-ELASTICITY APPLIED TO DAMS

Discussion

BY JOHN H. A. BRAHTZ, ESQ.

JOHN H. A. BRAHTZ,[41] ESQ. (by letter).[41a]—In closing this paper, the writer wishes to express his appreciation for the many interesting discussions that have appeared. All have been of a high caliber, and illustrate the fact that engineers are getting away from antiquated rules of thumb. With a fuller realization of the importance of two-dimensional stress problems, they are attacking such problems with open minds, more and more leaning toward the theory of elasticity and its co-partner, photo-elasticity. This inquisitive and open-minded attitude is decidedly encouraging. Only thus can the profession advance.

In reviewing the discussions, those pertaining to theory will be considered first.

Mr. Silverman refers to, and very clearly demonstrates, the usual method for obtaining solutions to the differential equation (Equations (3)) as a product of two functions, each of which is a function only of one of the independent variables (x, y) or (r, θ), respectively. As stated in discussing Equation (44) of the paper, this method was purposely not employed, in order to demonstrate the use of the general expression, Equation (4). In separating Equation (4) into its real and imaginary parts, both of which will be solutions to Equations (3), either Cartesian or polar co-ordinates may be used simply by placing the complex variable, z, equal to $x + i\,y$, or $r\,(\cos\theta + i\sin\theta)$, respectively. In fact, the stress components and displacements are often obtained most easily by carrying out the differentiations of the stress function in complex form before taking real or imaginary parts. In this connection,

NOTE.—The paper by John H. A. Brahtz, Esq., was published in September, 1935, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: November, 1935, by I. K. Silverman, Jun. Am. Soc. C. E.; December, 1935, by Fred L. Plummer, Assoc. M. Am. Soc. C. E.; January, 1936, by Messrs. A. G. Solakian and Lars R. Jorgensen; February, 1936, by Elmer O. Bergman, Assoc. M. Am. Soc. C. E.; and April, 1936, by D. P. Krynine, M. Am. Soc. C. E.

[41] With the U. S. Bureau of Reclamation, Denver, Colo.

[41a] Received by the Secretary April 23, 1936.

the following expressions are useful. Let $f = U + i\,V$ be a holomorphic complex function and consider the real part only; then:

$$\frac{\partial U}{\partial x} = \text{real part of } \left(\frac{df}{dz}\right) = \mathbf{R}\left(\frac{df}{dz}\right) \quad \ldots\ldots\ldots\ldots (109a)$$

$$\frac{\partial U}{\partial y} = \text{imaginary part of } -\left(\frac{df}{dz}\right) = -\,\mathbf{I}\left(\frac{df}{dz}\right) \ldots\ldots (109b)$$

$$\frac{\partial U}{\partial r} = +\,\mathbf{R}\left(e^{i\theta}\,\frac{df}{dz}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (109c)$$

and,

$$\frac{\partial U}{r\,\partial \theta} = -\,\mathbf{I}\left(e^{i\theta}\,\frac{df}{dz}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (109d)$$

Altogether, the theory of functions of a complex variable has proved extremely useful and expedient in two-dimensional stress analysis.

It is to be noted that the first function of Equation (4) is holomorphic, the last three are not, and must be treated as products of holomorphic functions and x, y, or r^2, respectively, in carrying out the differentiations.

It is again emphasized that many methods are available for finding solutions to Equations (3). The trick is to find the proper combination of solutions which satisfies the given boundary conditions. Unfortunately, no general method is available for this purpose. On the other hand, if such a solution has been found in some manner, it is fortunate to know that it is unique.

In other words, to find the solution for a given two-dimensional elastic problem it is necessary to try this or that combination of known biharmonics (those which satisfy Equations (3)). If the solution is finally obtained—often after days or weeks of hard labor—the writer generally must resort to a phrase, such as "consider the following function", and then merely prove that it does satisfy all the conditions of the problem without being able to explain just how he obtained the function. This often leaves the reader with the impression that the writer possesses some mysterious technique, which is not at all the case.

The value of the Airy stress function to the practical engineer begins with an intimate knowledge of the few known simple exact mathematical solutions and the ability to combine them in such a manner as to give a good approximate solution of the problem at hand. As a rule, an exact mathematical solution to a practical problem is so unwieldy as to be of little or no practical use. Indeed, the assumptions that must be made as to the physical properties and behavior of the materials involved in practical constructions are generally such as to preclude an exact solution of any practical problem. It should be understood that at best only a close estimate of the stresses in the actual structure can be expected even from the most careful mathematical analysis or photo-elastic experimentation.

The solutions and formulas presented by the writer hold strictly only under the assumptions stated in the "Introduction." In practice these assumptions are never completely fulfilled. In fact, in some instances, it is impossible ever to determine the real properties. This is especially true in the case of foundations.

Professor Krynine treats this subject in his excellent discussion and points out the various materials for which elastic statistical isotropy may be assumed and others in which it may not. This should serve as a valuable guide for the designer of foundations.

The writer has dealt only with elastic media, in plain stress or plain strain. Professor Krynine has made a valuable contribution in extending the theory of foundations to three-dimensional plastic media. The apparent ambiguity in the value of Poisson's ratio, μ_2, for the foundation treated by Professor Krynine ceases to exist when it is again remembered that the formulas only hold for elastic media in plain stress or strain. In that case, no assumption is made as to the incompressibility of soil particles.

In computing stresses in a semi-infinite (three-dimensional) solid due to its own weight, to a uniformly distributed surface loading, or to a combination of both loadings, it would be proper to specify the horizontal strains equal to zero. This gives rise to the formulas[42]:

$$\sigma_x = \sigma_z = K\,\sigma_y \ldots\ldots\ldots\ldots\ldots\ldots\ldots (110)$$

in which σ_y is the vertical stress equal to the total load above the point, and $K = \dfrac{\mu_2}{1 - \mu_2}$.

The displacement formulas presented by the writer are useful for the analysis of a long straight gravity dam on an elastic or quasi-elastic foundation, even if the elastic modulus of the dam itself differs from that of the foundation. The method of procedure of this problem was outlined under Case 2 in the paper. Numerical results have been developed as shown graphically in Fig. 20. The curves were obtained in connection with a study of Grand Coulee Dam (440 ft high), with an up-stream slope of 0.15, a down-stream slope of 0.8, and with concrete weighing 156 lb per cu ft. The curves indicate qualitatively the tendencies of distribution at the base as the foundation stiffens relative to the dam. The analysis did not take into consideration the stress concentrations near the heel and the toe. It is of interest to notice the general agreement between the curves for rigid foundation ($n = \infty$), and those obtained by Mr. B. F. Jakobsen[43].

Professor Bergman points out in a lucid manner the physical interpretation of the curvatures in the Airy surface. These properties are used in the "slab analogy", an experimental method by which stresses in a two-dimensional problem can be obtained by measuring curvatures and twists in a warped elastic slab. This is possible since the differential equation (Equations (3)), of the Airy elastic stress function, $\nabla^4 F = 0$, has the same form

[42] *Bulletin 117*, Iowa Eng. Experiment Station, pp. 39–43.

[43] *Transactions*, Am. Soc. C. E., Vol. 96 (1932), p. 489.

as the differential equation for the normal displacements, z. of a warped elastic slab, $\nabla^4 z = 0$, when the curvatures are held to a low magnitude. This means that the curvatures must be measured by means of microscopes or cor-

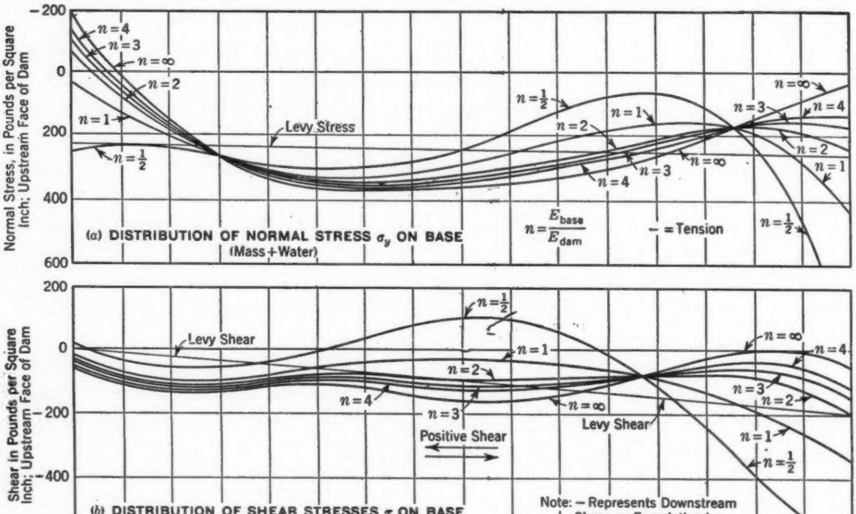

FIG. 20.—STRESS DISTRIBUTION, GRAND COULEE DAM.

respondingly sensitive measuring devices. In the analogy, the curvature of the slab at any point is proportional to the normal stress in a perpendicular direction in the prototype, and the twist at any point is proportional to the corresponding shear. Naturally, the slab must be warped by twists and curvatures at the boundaries that are themselves proportional to the boundary forces acting as the prototype.

Professor Bergman has been closely connected with model experiments conducted on dams by the U. S. Bureau of Reclamation, at the University of Colorado, under the direction of Ivan E. Houk, M. Am. Soc. C. E. It is indeed gratifying to contemplate the testimony of Professor Bergman that good general agreement exists between the theoretical results presented by the writer and this type of experimentation. That the stress concentrations at sharp corners were, to a large extent, relieved by plastic flow was to be expected. In deriving the approximate formulas for fillets the writer did so in order to be able to design a fillet with a safe elastic stress distribution without recourse to plastic time-flow of which very little is known under alternating load conditions.

In the "Introduction" to the paper the writer stated specifically that the intention was not to set up design criteria, and that the question of uplift was disregarded, except to assume that sufficient compression must exist to counteract the effect of uplift or liquid pore pressure. It is correct, therefore, to apply the theoretical results only to dam sections in which this is true, or

at least where the resulting tension is not sufficient to cause rupture. In connection with this question, Mr. Jorgensen cites several dam failures and shows that, in all cases, the total effective slope (up-stream plus down-stream slope) was far less than the slopes called for by the Levy criterion. No doubt, the question of internal liquid pressures in concrete is of great importance. Theoretical studies are now being made by the writer for the purpose of determining the effect of pore pressure on the stresses in hydraulic structures. No real headway can be made in application to practical structures until the question of "contact area" between particles has been settled experimentally for various concrete mixtures. Experiments to that end have been inaugurated at the laboratories of the Bureau of Reclamation. Some work has already been done in this field by various experimenters[44], but there seems to be a wide difference in opinion; in fact, the value given for effective contact area varies from 2 to 40 per cent. Until the question is settled beyond a doubt it would seem necessary, then, to assume the contact area to be practically zero, which means that full uplift must be assumed to exist. Assume, for example, that triangular uplift pressure exists at the base, equal to the full reservoir head at the heel and zero at the toe (which is reasonable for a nearly impervious dam on a pervious foundation), and apply this to the study of Grand Coulee Dam as shown in Fig. 6. It will be found that the vertical normal stresses (Fig. 6(*a*)) near the up-stream face and the horizontal normal stresses in the same region of the foundation, Fig. 6(*b*)) both become tension. The final section of Grand Coulee Dam has been designed with an up-stream slope of 0.15 and a down-stream slope of 0.8.

It will be noticed in Fig. 6(*a*) that the vertical normal compression stress near the heel is considerably less than that computed by the straight-line formula (Lévy). This would more than bear out Mr. Jorgensen's contention, and would indicate that, due to the restraint of the foundation, a section even heavier than the Levy cross-section is needed in cases of rather rigid foundations if the experiments confirm the evidence that full uplift must be applied to the dam. This contention is still more true if earthquake accelerations are to be considered and the situation is further aggravated by shrinkage stresses.

Fig. 20(*a*) would indicate that the ratio, n, between the elastic moduli in the foundation and the dam has an important influence on the stress situation at the heel. Further studies along this line should be made so that this seemingly important effect can be taken into consideration in the design of the minimum section on a given foundation. The approximations made in the analysis underlying Fig. 20 were such as to indicate that the curves for $n = 0.5$ deviate slightly too much from the Levy line.

A theoretical study of the situation around the heel, in which an open crack was assumed to exist under full uplift, demanded an effective total slope of 0.87 (with 150-lb concrete) for stability.

Mr. Jorgensen also touches upon the subject of curved gravity dams. The problem is three-dimensional, so that the individual case must be dealt with

[44] *Transactions*, Am. Soc. C. E., Vol. 99 (1934), p. 1052.

separately. The writer has no opinion to offer on the subject. Studies made by Mr. Jakobsen seem to justify the impression that the only point in favor of a curvature without arch action is public psychology.

Parts I and II of the paper were originally presented in two separate manuscripts. It was finally deemed advisable to combine them, and with the limitation set on a single paper, it became necessary to reduce both. Consequently, only sufficient description was retained of the photo-elastic material to show general methods and give the necessary checks on the theoretical results obtained in Part I. Furthermore, the photo-elastic work was conducted as early as 1931–1932 and much improved technique has been introduced since then. Reference to Fig. 21 will show that large spherical reflectors were used

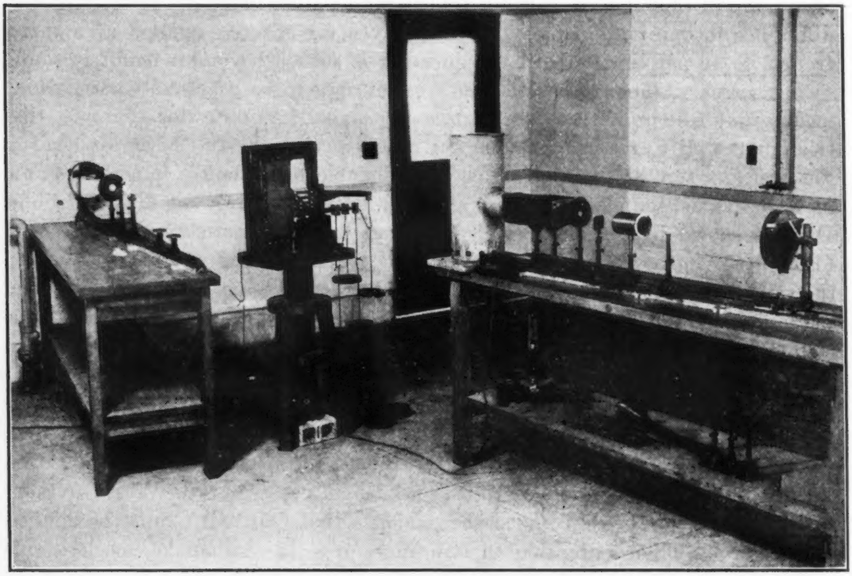

FIG. 21.—VIEW OF PHOTO-ELASTIC LABORATORY, CALIFORNIA INSTITUTE OF TECHNOLOGY.

instead of the much more expensive lenses, and that the polarizing unit is offset from the analyzing unit, thus providing a convenient large space for the model and operator[45].

The important features of the two discussions dealing with Part II are the methods for the determination of stresses due to body forces. It is of interest to note that Professor Plummer and Mr. Solakian have developed the centrifugal method independently. Professor Plummer finds that the method gives good results, but that it requires elaborate apparatus and very careful adjustments, and he prefers the direct-loading method described in his very excellent discussion and developed in co-operation with the U. S. Corps of Engineers, at Zanesville, Ohio. Professor Plummer gives new and valuable information in the preparation of gelatine models. If the optical creep in this very

[45] *Review of Scientific Instruments*, February, 1934, pp. 80–83.

optically sensitive substance will not prevent a reliable calibration, the writer believes it will be of inestimable value in photo-elastic experimentation, especially on foundations and related problems.

The writer has adopted the ingenious method, suggested by Dr. Biot, of loading the boundaries with normal force distributions equal to $K y'$, in which K is equal to the weight of the material in the prototype, and y' is the distance from the point of the boundary measured in the direction of the body force to a convenient line perpendicular to that force. This loading will deliver the correct isochromatics and isoclinics due to body forces.

Let the principal stresses obtained from the foregoing loading be σ'_1 and σ'_2; then, the actual stresses at a point due to body force will be.

$$\sigma_1 = \sigma'_1 - K y \quad \text{(111}a\text{)}$$

and,

$$\sigma_2 = \sigma'_2 - K y \quad \text{(111}b\text{)}$$

in which y is the ordinate to the point measured in the same system as y'. The correctness of this method can be proved directly by use of the Airy stress function. It has been used extensively and very successfully in the photo-elastic laboratory of the U. S. Bureau of Reclamation.

Mr. Solakian objects to the method of superposition of water loads and body forces. As a matter of fact, in the case of dams and many other civil engineering structures, both sets of stresses are usually required separately, either for design purposes or if strains are to be measured in the prototype due to water load (live load) only.

In conclusion, the writer wishes to announce the construction of a new photo-elastic interferometer for the Bureau of Reclamation by which the individual principal stresses at a point can be found directly. It will mark a new advance in technique which promises considerable time-saving. The apparatus was designed by John Soehrens, Jun. Am. Soc. C. E., in the photo-elastic laboratory, and was built under his supervision.

AMERICAN SOCIETY OF CIVIL ENGINEERS

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DISCUSSIONS

FLOOD AND EROSION CONTROL PROBLEMS AND THEIR SOLUTION

Discussion

BY MESSRS. DONALD M. BAKER, AND E. COURTLANDT EATON

DONALD M. BAKER,[27] M. AM. SOC. C. E. (by letter).[27a]—The flood and erosion control problem in Los Angeles County is complex in the extreme. It may be considered from the following aspects: Physical, meteorological, hydrological, recreational, cultural, historical, financial, and political.

Physical.—Settlement occurs principally upon the coastal plain, 1 000 sq miles in extent, lying between the Pacific Ocean and the base of the San Gabriel Range, where the elevation reaches about 1 000 ft. The mountains then rise abruptly within a short distance to elevations of from 6 000 to 10 000 ft, their slopes covered with a deep soil, easily eroded, which supports a cover of heavy brush and timber. This cover constitutes a serious fire hazard, and when destroped, heavy flood run-off and débris flows result. Slopes of streams draining the south side of the range are steep, and available reservoir sites are of small capacity, while the geologic structure of the range affords very unsatisfactory foundation conditions. About half way between the San Gabriel Range and the ocean, and extending parallel to it, a series of low hills traverse the coastal plain, with two openings, through which pass the Los Angeles and San Gabriel Rivers. Three large ground-water basins—the San Gabriel Valley, San Fernando Valley, and the coastal plain—supplied by percolation of streams draining the mountains, form valuable sources of water supply for overlying lands.

Meteorological.—Mean precipitation, concentrated principally in the period from December to March, inclusive, increases from 10 to 12 in. near the ocean to 20 to 25 in. at the base of the mountains, and to 35 in, and more at their crest, with annual variations of from 40% to 250% of the mean, and cyclic fluctuations extending over long periods.

NOTE.—The paper by E. Courtlandt Eaton, M. Am. Soc. C. E., was published in September, 1935, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: November, 1935, by Messrs. Arthur G. Pickett, and R. W. Davenport; December, 1935, by C. S. Jarvis, M. Am. Soc. C. E.; February, 1936, by Messrs. Harry F. Blaney, W. P. Rowe, and J. B. Lippincott; and March, 1936, by Messrs. E. I. Kotok and C. J. Kraebel.

[27] Cons. Engr., Los Angeles, Calif.

[27a] Received by tne Secretary March 6, 1936.

The long dry season extending from the spring until the beginning of winter, with low humidity which continues for weeks at a time, creates a condition in the brush and tree cover which makes it very susceptible to fire.

Hydrological.—Damaging floods of county-wide extent have occurred singly, or in groups, at intervals of about twenty-five years. Unit rates of run-off from such floods are high.

Recreational.—Proximity of the mountainous areas to the large population on the coastal plain invites use of the mountains for recreation, which, in turn, requires that they be made accessible. This has been done through the construction of many trails and, recently, of automobile roads. This enhanced accessibility has greatly increased the fire hazard with its threat of removing protective brush cover, and many serious fires have occurred, which have further added to the flood run-off and erosion.

Cultural.—The population of the coastal plain has increased from 500 000 in 1916, when the last major flood occurred, to about 2 250 000 in 1936, or 350 per cent. Mr. Eaton's estimate that only 8% of the present population has experienced or has a realization of a major flood is evidence of the difficulty met in obtaining public interest in the problem. Much settlement and development have taken place on areas subject to flood menace without an appreciation of the menace either on the part of the developers, settlers, or of public officials. The average population density of the area is in excess of 2 000 persons per sq mile, and high values of land and improvements exist, many of such values occurring in areas subject to flood menace.

Historical.—Local attention was focused upon flood-control problems following the 1914 flood, the first to cause serious damage. Major floods occurring previous to that date, in the 1880–90 decade, although greater in magnitude than those of 1914 and 1916, caused far less damage because of the relatively small population and the lack of intensive development. Following 1914, a flood-control district was formed, embracing practically the entire County south of the mountains. The plan of flood control first adopted consisted primarily of channel rectifications and improvement with little attention given to water conservation. A bond issue of about $4 500 000 was voted in 1917 to carry out this plan.

The great increase in population occurring in the 1920–30 decade focused attention upon the need for conservation of local water supplies, and, in 1924, a plan, hastily conceived and prepared without adequate hydrologic and other data, was devised for which a bond issue of $35 000 000 was voted. In this plan conservation of water was emphasized, the bulk of expenditure being for storage dams in the mountains, foremost of which was a gigantic concrete dam in the San Gabriel Canyon estimated to cost $25 000 000. Some of the smaller dams proposed in this plan were built. The large dam in the San Gabriel Canyon was abandoned, due to poor foundation conditions, and plans for two smaller dams with much less storage capacity were substituted. With the advent of a bountiful supply of water from the Colorado River, interest in conservation of flood water has waned, and the present construction program, amounting to nearly $20 000 000 and carried on by the Corps of Engi-

neers, U. S. Army, through the assistance of a large Federal grant, is again directed entirely toward channel rectification and improvement.

Financial.—The cost of the work done by the Flood Control District is met by a district tax upon land and improvements only. Since the assessed valuation of the City of Los Angeles, which is within the District, has ranged from 60% to 70% of that of the entire District, the City has been in the past (and will be in the future) called upon to meet this percentage of the total cost of the program, except that done under Federal grant. Except for some channel improvement in the Los Angeles River and the Pacoima and Tujunga Dams, the bulk of the money spent for construction work has been on structures and projects outside the city limits, although under the present program involving Federal aid, a considerable portion of the funds are allocated to projects within the City of Los Angeles. The latter has a very serious storm-water problem within its boundaries, and until a few years ago it had made large annual expenditures for the construction of storm drains, the costs being raised by local assessment.

Political.—The Board of Supervisors of Los Angeles County acts as the legislative body of the Los Angeles County Flood Control District, and various other County officials, such as the Assessor, Tax Collector, Auditor, County Counsel, also act for the District. The latter has its own engineering personnel which, however, since the District is a separate political entity, is not subject to County Civil Service regulations. This form of organization, originally proposed in the interest of economy, has not been satisfactory, as flood-control administration and policy have been submerged in general County issues, and popular opinion has not found an opportunity to express itself on flood control alone.

Proper control measures group themselves into the following:

(1) Water-shed management, including the preservation of existing vegetal cover, renewal of destroyed cover, and prevention of erosion;

(2) Disposition or conservation of flood waters (or both disposition and conservation), including construction of regulation and storage reservoirs, spreading works, and flood channels; and,

(3) Control of land utilization in flood-menaced areas, including the reservation of land for reservoir and channel easements and the control of land utilization and improvement in menaced areas.

Recently, much excellent work has been accomplished by Federal and County forestry agencies in water-shed management. Motor roads to afford accessibility for fire-fighting have been constructed, and equipment and organization for such fighting have been provided. Work is progressing in restoration of burnt-over areas and in prevention of erosion from bare slopes. An extensive program for collecting basic hydrologic data, lacking at the time previous plans were made and bond issues voted, has been under way for some years past.

An attempt was made in the 1935 Session of the State Legislature to amend the Flood Control Act, creating for the Flood Control District a legislative body separate from the County Board of Supervisors, providing for

the preparation and adoption of a comprehensive plan of flood control before any further construction work was done, and placing the personnel of the District's staff under civil service. Although such revision had the hearty support of all civic bodies interested in the subject, political opposition in certain quarters prevented its accomplishment.

Widespread differences of opinion exist among those hoping to benefit by the conservation features of the District's activities as to the title to the water which will be developed by such features, and until this subject is settled and a policy adopted governing the utilization of such water, a threat of extensive litigation will continue.

Nothing so far has been done toward the control of land utilization in flood-menaced areas, although it would appear that, once such areas were designated after proper study and investigation, legal authority probably exists to prevent such development and utilization through regulation of the sub-division of land by the planning commissions of the County and of the various cities within the District.

Neither has any accomplishment been had in the development of a comprehensive plan of flood and erosion control for the District in which all the various phases and aspects of the problem—regulation, conservation, land use, and financing—are considered and co-ordinated. Until such a comprehensive plan has been developed and adopted and the opportunistic and drifting policy which has characterized past efforts to solve the problem has been abandoned, future occurrence of events which are related by the author of this paper may be expected.

E. Courtlandt Eaton,[28] M. Am. Soc. C. E. (by letter).[28a]—Mr. Pickett states correctly that fire-prevention methods on water-sheds are the cheapest and best insurance against the débris menace. Next in order of economy are provisions for quick access to incipient fires and provisions for adequate supplies of water to assist the fire fighters. The quantity of water required is not large since modern "fog nozzles", or high-pressure nozzles creating a fine spray as a protection to men and to increase the humidity locally, have been found to be very effective. In line with the plan advanced by Mr. Pickett, it has been proposed to build submerged and covered concrete tanks in the mountains at accessible locations which may be filled from rainfall, by collector channels, or by adjacent paved areas, in readiness for the fighting of possible fires during a succeeding dry season. As stated by Mr. Pickett, limited appropriations have prevented fulfillment of these plans.

If measures for prevention and suppression fail, débris basins must be resorted to as the last line of defense at costs which may reach capital expenditures as great as $150 000 per sq. mile of water-shed.

Mr. Davenport mentions the accepted procedure, in determining annual run-off quantities, of considering run-off as a residual after subtracting evaporation, transpiration, and other losses. This method is entirely proper, of course, where the primary purpose is to establish a relation between rain-

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[28] Cons. Engr., Los Angeles, Calif.

[28a] Received by the Secretary April 25, 1936.

fall and run-off, and are particularly applicable to localities where rainfall is less flashy in character and water-sheds are not so subject to physical changes as is the case in Los Angeles County.

Under Los Angeles County Weather and physical conditions a given quantity of rainfall in a season may produce widely varying amounts of run-off, depending upon the intensities and the previous condition of saturation of the water-shed, and detail study of daily and hourly precipitation must be made. In the studies of Big Tujunga water-shed the problem was mainly that of arriving at the regulating capacity required and the expected peak flows during the critical 4-day period rather than to determine the seasonal run-off. Table 5, therefore, is merely a general and preliminary study based upon the indices of seasonal wetness method and was used as a guide in the selection of a critical season for detailed analysis. It is possible that the total run-off for that season, under normal water-shed conditions, would be close to 250 000 acre-ft. However, there are historical records indicating that the water-shed had not fully recovered from a previous fire so that the actual run-off may have been even greater than that shown in Table 5. Since all historical and rainfall records pointed to 1884 as the year of major flood in a 50-yr period, that year was selected for detailed analysis.

The details of the method used in computing the 4-day peak flows are too long for reproduction herein, but in brief the procedure included: (1) The measurement by planimeter, of sub-areas from a map similar to Fig. 5; (2) the distance from the point of delivery to the next succeeding point down stream was measured and the channel slope computed; (3) times of concentration were determined for the sub-areas; and (4) a coefficient of run-off was selected for the average typical soil and cover characteristics as determined by field investigations.

The flow was then computed from the rational expression, $Q = C\ I\ A$, in which Q = run-off in cubic feet per second; C = coefficient of run-off; I = average intensity of rainfall during the time of concentration of the sub-areas, in inches per hour; and, A = area of sub-area, in acres.

The relation between the daily peaks in the maximum 4-day flood period was derived from co-related studies of the San Gabriel River water-shed on which there are considerably longer rainfall and run-off records than on the Big Tujunga water-shed.

Mr. Jarvis states that the conditions in Los Angeles County are extreme and that the County has a combination of physical features and high property values not commonly met elsewhere. The writer agrees. Nevertheless, similar erosion problems to a lesser degree exist elsewhere in the West although, fortunately, high concentration of population and property values do not as yet lie within the paths of débris flows.

Mr. Blaney mentions the excellent experimental work done by Mr. Mitchelson, of the U. S. Bureau of Agricultural Engineering, from which much valuable information has been obtained. The relatively high rates of percolation obtained by Mr. Mitchelson as compared with those of the writer in Table 12 are attributable to the relatively clear water available for Mr.

Mitchelson's experiments. The percolation rates given by the writer are in the main those during receding flood-flow periods when the water was cloudy. Were it possible, in spreading operations, to count upon handling a reasonably clear stream flow the "basin" method of spreading could be used at a considerable saving in land costs and spreading works. Under conditions where the water carries considerable suspended matter, the ditch system, in the writer's judgment, is preferable.

Mr. Rowe infers that the coincidence of a heavy rain following a fire is remote. The writer cannot agree. A storm of high intensity, coming any time within a 3-yr, or more, regrowth period after a fire will cause a débris flow; in fact, a second high-intensity storm within that period may produce even a greater débris flow than occurred with the first storm due to the previously loosened water-shed condition. Mr. Rowe mentions check dams as a possible indirect source of some of the débris. A thorough study made of the canyons immediately following the débris flows does not support his theory. Mr. Rowe also attributes a portion of the débris flow in Haines Canyon débris basin to the operation of a gravel pit which formed the nucleous of the aforementioned basin. This theory is borne out neither by the examinations made of the water-shed to trace the sources of débris, nor by the analyses of débris in the basin, much of which consisted of earth and top-soil originating from water-shed slopes.

Mr. Lippincott mentions one of the difficulties under which the Los Angeles County Flood Control District has labored. Although there are in the District's files and on file as public records, many complete reports containing valuable engineering data, legal restrictions have prevented the financing of their reproduction for general distribution and a lack of general knowledge other than that contained in the daily press has resulted.

Mr. Lippincott mentions the publication of a "Rainfall and Runoff Report, Seasons 1932–33 and 1933–34." In fairness to previous administrations many rainfall data were obtained extending back almost as far as the formation of the District and complete rainfall-run-off reports in the aforementioned form, have been issued each year since 1927.

Mr. Lippincott draws a comparison in Table 15 between the four dams of highest unit cost of the twelve constructed by the District. The average cost of all dams built by the District has been about \$200 per acre-ft, ranging from \$35 to \$1 300 per acre-ft. The average is equivalent to about 13 cents per cu yd of débris capacity.

Undoubtedly, dams, even at the highest cost mentioned, are more economical than débris basins. However, dam sites are limited and the débris basin takes its place where no adequate dam sites are available.

Incidentally, considering flood control and conservation features, a statement of cost per acre-foot of storage capacity alone is not indicative of flood-regulating values, since the reservoirs may regulate in a single season amounts three to four times their nominal storage capacity and the conservation or hold-over storage features are present in the underground basins fed by the regulated flows.

Messrs. Kotok and Kraebel have mentioned the work being done under their direction by the California Forest and Range Experiment Station of the U. S. Forest Service. From their results will be obtained invaluable quantitative data that will furnish a basis for future planning of remedial and preventive measures. The writer agrees that his value of 1 500 cu yd of débris per sq mile annually from unburned water-sheds may be high.

Fig. 15 shows graphically the difficulty in estimating rates of débris from the high-water mark, *A*, and taking the cross-section, *C*, after the recession of the flood. The section at the time of maximum flow might be at any place between *B* and *A*.

Mr. Baker mentions the difficulty in obtaining public interest in flood-protection problems. Fundamentally, the Flood-Control Act itself is responsible. Under the Act, the members of the County Board of Supervisors, only a fraction of whose time can be given to flood-control matters, act as directors without additional salary for that service. Furthermore, the Act provides that the County Counsel, County Auditor, and County Purchasing Agent act in their several capacities for the District without additional compensation for this service. Each of these report directly to the Board of Supervisors. Thus, there is no semblance of centralized control or responsibility. This condition, in conjunction with the failure to finance, legally, the general dissemination of reports and information, or the lack of legal authority to finance a public relations bureau has placed the District in a defensive position.

Although only partly completed as a completely connected unit, the works built thus far have safeguarded thousands of lives and have prevented, and are preventing, destruction to property that might otherwise mount to sums several times the expenditures made for protection.

Incidentally, a Comprehensive Plan Report was issued by the District in 1931, drawing attention to the menaced areas and outlining needed protective works. As stated by Mr. Baker attempts to amend the Flood Control Act have so far been unsuccessful.

The writer is much gratified for the constructive discussions that have been presented on his paper.

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DISCUSSIONS

TAPERED STRUCTURAL MEMBERS: AN ANALYTICAL TREATMENT

Discussion

BY MESSRS. C. W. DUNHAM, FANG-YIN TSAI, A. A. EREMIN, AND AUSTIN H. REEVES

C. W. DUNHAM,[32] M. AM. SOC. C. E. (by letter).[32a]—The method of analysis set forth in this paper should be useful as a means of checking design calculations for certain special structures after the sections and stresses have been determined by other simple and approximate methods. However, there is one point or feature which is touched upon almost too lightly; namely, the "working lines."

One of the most common uses of the tapered section is in rigid frames and the location of the working lines may cause considerable variation in the results of the analysis. It may be advantageous for the authors to give more emphasis to this matter and to include in their method the necessary data to serve as a guide of sufficient accuracy so that the possible error which may be introduced, due to incorrect working lines, shall not exceed that which would result from the use of more simple methods of analysis.

In one rather extreme case of rigid frames, the stresses resulting from the use of a working line through the center of gravity of the center section varied more than 10% from those computed by using a working line through the center of gravity of the haunch. Of course, this error works in opposite directions as far as the stresses at the crown and haunch are concerned, but, in any event, it is too large to neglect.

Although the rigid frame is only one of the types of structures in which the tapered member is used, it is an important type. Such frames often have a definite camber or curve in the top chord. This adds a thrust in the frame, because of the tendency toward a partial arch action which is not included in the formulas given in the paper, but which should not be omitted in the calcu-

NOTE.—The paper by Walter H. Weiskopf and John W. Pickworth, Assoc. Members, Am. Soc. C. E., was published in October, 1935, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: February, 1936, by Messrs. Fred L. Plummer, and LeRoy W. Clark; and March, 1936, by Messrs. E. G. Paulet, J. Charles Rathbun, and Halvard W. Birkeland.

[32] Asst. Engr., Design Div., Port of New York Authority, New York, N. Y.

[32a] Received by the Secretary January 27, 1936.

lations. Furthermore, assuming a portal-like member supported by columns in a building, as suggested by Fig. 12, the location of the working line in the tapered section may greatly affect the moments in the columns themselves.

FANG-YIN TSAI,[33] ASSOC. M. AM. SOC. C. E. (by letter).[33a]—The outstanding feature of the method described in this paper is the substitution of an approximate I-curve for the actual one so that the various constants can be expressed by formulas derived by direct integration. Several sets of such formulas, most of which are rather lengthy and involve fractional exponents, are presented for the computation of the various constants involved in the moment distribution method proposed by Hardy Cross, M. Am. Soc. C. E. To those who prefer to solve the problem by formulas, this paper may be of value.

In the aforementioned formulas, the authors have introduced the various constants, F_1, F_2, F_3, etc., without assigning to them any physical meaning. Elsewhere[34], the writer has stated that,

"* * * to analyze, by any method, a continuous structure with a moment of inertia varying in any manner, five independent constants or coefficients must be known (three beam coefficients and two load coefficients) for every span of the structure considered as a simply supported beam; and these five coefficients may be expressed in various ways to suit any particular method of analysis."

It may be of interest to compare the authors' F-constants with those of the other systems, and, before doing so, a brief explanation of the latter with regard to their characteristics and adaptability to certain methods of analysis will be necessary. Among the various prevailing systems of expressing these beam and load constants, the following are the most outstanding: (1) The method of angle changes; (2) the method of moment areas; (3) application of Ruppel's constants; (4) application of Strassner's constants; and (5) application of constants in the Cross method.

(1).—Angle Changes.—The system of expressing the constants in terms of angle changes was first introduced apparently, by German authors and was used by Ernst Suter[35] and A. Strassner[36] in developing their methods of analysis. Considering the beam, LR, in Fig. 22, as simply supported and subjected to any loading, the following angle changes occur: α^0 = angle change due to the given loading (see Fig. 22(b)); α = angle change due to a moment of unity applied at the same end as α^0 (see Fig. 22(c) and Fig. 22(d)); and β = angle change at either end of a beam due to a moment of unity applied at the other end (see Fig. 22(c) and Fig. 22(d)). The subscripts, L and R, denote left end and right end, respectively.

(2).—Moment Areas.—Again, considering Beam LR (Fig. 22) as simply supported and subjected to any loading[34]: A_0 = area of the $\frac{M}{I}$-diagram due

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[33a] Received by the Secretary January 21, 1936.

[34] *Proceedings*, Am. Soc. C. E., May, 1935, p. 692.

[35] "Die Methode der Festpunkte", Julius Springer, Berlin, 1932.

[36] "Neuere Methoden zur Statik der Rahmenträgwerke und der Elastischen Bogenträger", Vol. 1, Wilhelm Ernst & Sohn, Berlin, 1925.

to loading; g = abscissa ratio of the centroid of A_0 from End L; A = area of the $\frac{M}{I}$-diagram due to a moment of unity applied at the designated end (the end indicated by the subscripts, L or R); and u and v = abscissa ratios, of the centroid of A_L and A_R from Ends L and R, respectively.

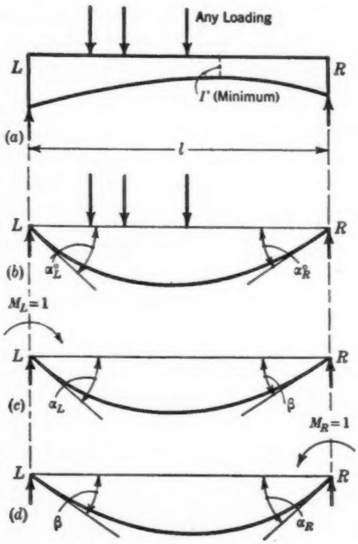

FIG. 22.

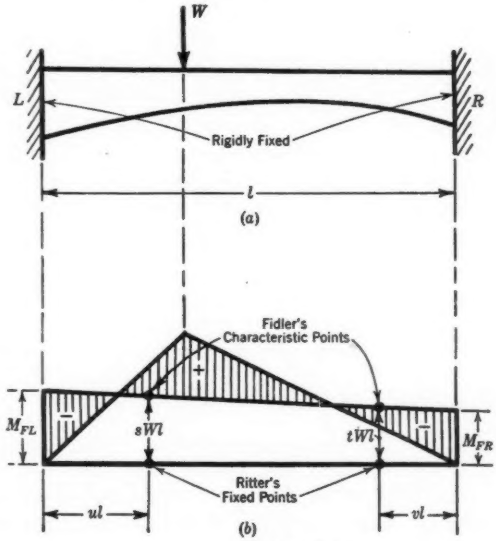

FIG. 23.

Of the six constants, only five are independent, since it can be shown easily that, according to Maxwell's principle of reciprocal deflections (angular), the following relation is always valid:

$$A_L u = A_R v \text{..........................} (167)$$

(3).—Constants in Ruppel's Tables[37].—The tables introduced by Walter Ruppel, Assoc. M. Am. Soc. C. E., were converted and also extended from Strassner's tables[36] for use in the graphical method of fixed points or conjugate points developed by T. C. Fidler[38], A. Ostenfeld[39], and L. H. Nishkian and D. B. Steinman[40], Members, Am. Soc. C. E. In these tables are found the beam constants or coefficients, p, q, u, and v, and the load coefficients, s and t. The constants, u and v, are identical with those of the moment areas previously mentioned. The physical meaning of the constants, u, v, s, and t, is shown in Fig. 23. Again, only five of the six constants are independent, since the following relationship is always valid:

$$p u = q v \text{..........................} (168)$$

[37] *Transactions*, Am. Soc. C. E., Vol. 90 (1927), pp. 167–187.
[38] *Minutes of Proceedings*, Inst. C. E., London, Vol. 74 (1883), p. 196.
[39] "Teknisk Statik" (in Danish), Vol. II, Copenhagen, 1925, pp. 83–142.
[40] *Transactions*, Am. Soc. C. E., Vol. 90 (1927), pp. 1–48.

(4).—Constants in Strassner's Tables[36].—The beam constants[41], ϕ_{aL}, ϕ_{aR}, and ϕ_{β}, and the load constants, ϕ_s and ϕ_t, are presented in Strassner's tables. An important feature of this system is that, for a uniform load over the full length of a symmetrical double-tapered member of any shape, the load constants, ϕ_s and ϕ_t, are always equal to unity. Hence, for the given case, no tables are needed for those two constants.

(5).—Constants in Cross' Method of Moment Distribution.—In applying the method of moment distribution developed by Professor Cross[42], the following constants for every span of a continuous structure must be first determined: C_{LR} = the factor required to carry over the moment at End L, to End R. (C_{RL} = the factor required to carry over the moment at End R, to End L. The sign for C_{LR} and C_{RL} will be considered here as positive); S_L = the stiffness factor when a moment is applied at End L with End R fixed. (S_R refers to the moment applied at End R with End L fixed. This stiffness factor is in accordance with Professor Cross' definition[43]. That given by the authors' in Equation (61) or Equation (62) is equal to $\frac{S}{4E}$, which corresponds to $\frac{I}{l}$ for beams of constant moment of inertia. Similarly, the modified stiffness factor given by the authors' in Equation (110) or Equation (111) is equal to $\frac{S'}{4E}$, which corresponds to $\frac{3I}{4l}$ for beams of constant moment of inertia); and M_{FL} = the fixed-end moment at End L, due to loading, with both ends fixed. Again, only five of the six constants are independent since the following relation is always valid:

$$C_{LR}\,S_L = C_{RL}\,S_R \qquad (169)$$

In addition, the following modified constants are worthy of note: S'_L = the stiffness factor when the moment is applied at End L with the other end simply supported; and M'_{FL} = the fixed-end moment at End L due to the given loading, with the other end simply supported. The following relations can be found easily:

$$S'_L = S_L\,(1 - C_{LR}\,C_{RL}) \qquad (170)$$

$$S'_R = S_R\,(1 - C_{LR}\,C_{RL}) \qquad (171)$$

$$M'_{FL} = M_{FL} + C_{RL}\,M_{FR} \qquad (172)$$

$$M'_{FR} = M_{FR} + C_{LR}\,M_{FL} \qquad (173)$$

and,

$$C_{LR}\,S'_L = C_{RL}\,S'_R \qquad (174)$$

[41] There are no such notations in Strassner's tables ("Neuere Methoden zur Statik der Rahmenträgwerke und der Elästischen Bogenträger", Vol. 1, pp. 101–112), and they are introduced by the writer, indicating that they are the functions of a_L, a_R, β, s, and t. Similar tables are also found in "Die Methode der Festpunkte", by Ernst Suter, pp. 413–419.

[42] *Transactions*, Am. Soc. C. E., Vol. 96 (1932), p. 1.

[43] *Loc. cit.*, p. 2.

TABLE 3.—Conversion of Beam and Load Constants in the Various Systems*

Units	Angle changes	Moment areas	Ruppel	Strassner	Cross
Radians / Kip-feet	α_L	$\frac{A_L}{E}(1-u)$	$\frac{p\,l}{EI'}(1-u)$	$\frac{l}{3\,EI'}\phi_{\alpha_L}$	$\frac{1}{S_L(1-C_{LR}C_{RL})}$
	α_R	$\frac{A_R}{E}(1-v)$	$\frac{q\,l}{EI'}(1-v)$	$\frac{l}{3\,EI'}\phi_{\alpha_R}$	$\frac{1}{S_R(1-C_{LR}C_{RL})}$
	β	$\frac{A_L u}{E}$ or $\frac{A_R v}{E}$	$\frac{pul}{EI'}$ or $\frac{qvl}{EI'}$	$\frac{l}{6\,EI'}\phi_\beta$	$\frac{C_{RL}}{S_L(1-C_{LR}C_{RL})}$ or $\frac{C_{LR}}{S_R(1-C_{LR}C_{RL})}$
Radians	α°_L	$\frac{A_0}{E}(1-g)$	$\frac{Wl^2}{EI'}sp$	$\frac{Wl^2}{K\,6\,EI'}\phi_\beta\,\phi_s$	$\frac{M_{FL}+C_{RL}M_{FR}}{S_L(1-C_{LR}C_{RL})}$
	α°_R	$\frac{A_0}{E}g$	$\frac{Wl^2}{EI'}tq$	$\frac{Wl^2}{K\,6\,EI'}\phi_\beta\,\phi_t$	$\frac{M_{FR}+C_{LR}M_{FL}}{S_R(1-C_{LR}C_{RL})}$
Ratio / Cubic feet	$E(\alpha_L+\beta)$	A_L (NOTE.— $A_L u = A_R v$)	$\frac{pl}{I'}$	$\frac{l}{6\,I'}(2\phi_{\alpha_L}+\phi_\beta)$	$\frac{E(1+C_{RL})}{S_L(1-C_{LR}C_{RL})}$
	$E(\alpha_R+\beta)$	A_R	$\frac{ql}{I'}$	$\frac{l}{6\,I'}(2\phi_{\alpha_R}+\phi_\beta)$	$\frac{E(1+C_{LR})}{S_R(1-C_{LR}C_{RL})}$
Kips / Sq. ft.	$E(\alpha^\circ_L+\alpha^\circ_R)$	A_0	$\frac{Wl^2}{I'}(sp+tq)$	$\frac{Wl^2}{K\,6I'}\phi_\beta(\phi_s+\phi_t)$	$E[M_{FL}S_R(1+C_{RL})+M_{FR}S_L(1+C_{LR})]\div S_L S_R(1-C_{LR}C_{RL})$
Ratio	$\frac{\beta}{\alpha_L+\beta}$	u	u	$\frac{\phi_\beta}{2\phi_{\alpha_L}+\phi_\beta}$	$\frac{C_{RL}}{1+C_{RL}}$
	$\frac{\beta}{\alpha_R+\beta}$	v	v	$\frac{\phi_\beta}{2\phi_{\alpha_R}+\phi_\beta}$	$\frac{C_{RL}}{1+C_{LR}}$
	$\frac{\alpha^\circ_R}{\alpha^\circ_L+\alpha^\circ_R}$	g	$\frac{tg}{sp+tq}$	$\frac{\phi_t}{\phi_s+\phi_t}$	$\frac{1}{1+\frac{S_R(M_{FL}+C_{RL}M_{FR})}{S_L(M_{FR}+C_{LR}M_{FL})}}$
Ratio	$\frac{EI'}{l}(\alpha_L+\beta)$	$\frac{A_L I'}{l}$	p (NOTE.— $pu = qv$)	$\frac{2\phi_{\alpha_L}+\phi_\beta}{6}$	$\frac{EI'}{l}\,\frac{(1+C_{RL})}{S_L(1-C_{LR}C_{RL})}$
	$\frac{EI'}{l}(\alpha_R+\beta)$	$\frac{A_R I'}{l}$	q	$\frac{2\phi_{\alpha_R}+\phi_\beta}{6}$	$\frac{EI'}{l}\,\frac{(1+C_{LR})}{S_R(1-C_{LR}C_{RL})}$
	$\frac{\alpha^\circ_L}{Wl(\alpha_L+\beta)}$	$\frac{A_0(1-g)}{A_L Wl}$	s	$\frac{\phi_\beta\,\phi_s}{K(2\phi_{\alpha_L}+\phi_\beta)}$	$\frac{M_{FL}+C_{RL}M_{FR}}{Wl(1+C_{RL})}$
	$\frac{\alpha^\circ_R}{Wl(\alpha_R+\beta)}$	$\frac{A_0\,g}{A_R Wl}$	t	$\frac{\phi_\beta\,\phi_t}{K(2\phi_{\alpha_R}+\phi_\beta)}$	$\frac{M_{FR}+C_{LR}M_{FR}}{Wl(1+C_{LR})}$
Ratio	$\frac{3\,EI'}{l}\alpha_L$	$\frac{3I'A_L(1-u)}{l}$	$3p(1-u)$	ϕ_{α_L}	$\frac{3\,EI'}{lS_L(1-C_{LR}C_{RL})}$
	$\frac{3\,EI'}{l}\alpha_R$	$\frac{3I'A_R(1-v)}{l}$	$3q(1-v)$	ϕ_{α_R}	$\frac{3\,EI'}{lS_R(1-C_{LR}C_{RL})}$
	$\frac{6\,EI'}{l}\beta$	$\frac{6I'A_L u}{l}$ or $\frac{6I'A_R v}{l}$	$6pu = 6qv$	ϕ_β	$\frac{6\,EI'C_{RL}}{lS_L(1-C_{LR}C_{RL})}$ or $\frac{6\,EI'C_{LR}}{lS_R(1-C_{LR}C_{RL})}$
	$\frac{K\alpha^\circ_L}{Wl\beta}$	$\frac{KA_0(1-g)}{A_L uWl}$ or $\frac{KA_0(1-g)}{A_R vWl}$	$\frac{Ks}{u}$	ϕ_s	$\frac{K}{Wl}\times\frac{M_{FL}+C_{RL}M_{FR}}{C_{RL}}$
	$\frac{K\alpha^\circ_R}{Wl\beta}$	$\frac{KA_0 g}{A_L uWl}$ or $\frac{KA_0 g}{A_R vWl}$	$\frac{Kt}{v}$	ϕ_t	$\frac{K}{Wl}\times\frac{M_{FR}+C_{LR}M_{FL}}{C_{LR}}$
Ratio	$\frac{\beta}{\alpha_R}$	$\frac{v}{1-v}$	$\frac{v}{1-v}$	$\frac{\phi_\beta}{2\phi_{\alpha_R}}$	C_{LR}
	$\frac{\beta}{\alpha_L}$	$\frac{u}{1-u}$	$\frac{u}{1-u}$	$\frac{\phi_\beta}{2\phi_{\alpha_L}}$	C_{RL}
Kip-feet	$\frac{\alpha_R}{\alpha_L\alpha_R-\beta^2}$	$\frac{E(1-v)}{A_L(1-u-v)}$	$\frac{EI'(1-v)}{pl(1-u-v)}$	$\frac{3\,EI'\phi_{\alpha_R}}{l(\phi_{\alpha_L}\phi_{\alpha_R}-\frac{1}{4}\phi^2_\beta)}$	S_L
	$\frac{\alpha_L}{\alpha_L\alpha_R-\beta^2}$	$\frac{E(1-u)}{A_R(1-u-v)}$	$\frac{EI'(1-u)}{ql(1-u-v)}$	$\frac{3\,EI'\phi_{\alpha_L}}{l(\phi_{\alpha_L}\phi_{\alpha_R}-\frac{1}{4}\phi^2_\beta)}$	S_R
	$\frac{\alpha_R\alpha^\circ_L-\beta\alpha^\circ_R}{\alpha_L\alpha_R-\beta^2}$	$\frac{A_0[(1-g)-v]}{A_L(1-u-v)}$	$\frac{Wl[s-(sv+tu)]}{(1-u-v)}$	$\frac{Wl}{2K}\times\frac{\phi_\beta(\phi_{\alpha_R}\phi_s-\frac{1}{2}\phi_\beta\phi_t)}{(\phi_{\alpha_L}\phi_{\alpha_R}-\frac{1}{4}\phi^2_\beta)}$	M_{FL}
	$\frac{\alpha_L\alpha^\circ_R-\beta\alpha^\circ_L}{\alpha_L\alpha_R-\beta^2}$	$\frac{A_0[g-u]}{A_R(1-u-v)}$	$\frac{Wl[t-(sv+tu)]}{(1-u-v)}$	$\frac{Wl}{2K}\times\frac{\phi_\beta(\phi_{\alpha_L}\phi_t-\frac{1}{2}\phi_\beta\phi_s)}{(\phi_{\alpha_L}\phi_{\alpha_R}-\frac{1}{4}\phi^2_\beta)}$	M_{FR}

NOTE. — $C_{LR}S_L = C_{RL}S_R$; $S'_L = S_L(1 - C_{LR}C_{RL})$; $S'_R = S_R(1 - C_{LR}C_{RL})$; $M'_{FL} = M_{FL} + C_{RL}M_{FR}$; $M'_{FR} = M_{FR} + C_{LR}M_{FL}$; $C_{LR}S'_L = C_{RL}S'_R$

* Signs for the various quantities in the table are disregarded; 1 kip = 1 000 lb.

Comments.—The relation between all the constants of the aforementioned five systems are presented in Table 3[44], in which all the quantities on the same horizontal line are equal to one another and the additional notation is defined as follows: I' = minimum moment of inertia of the tapered member, in feet^4; E = modulus of elasticity, in kips per square foot; l = span length, in feet; W = total load on the span, in kips (in applying Strassner's and Ruppel's tables, the load constants for each load must be computed separately and used with the proper summation if there are several loads on the span); and K = a factor to be used in connection with Strassner's tables only. $K = 1$ for concentrated load and $K = 4$ for uniform load over the full span length. By means of this conversion table (Table 3), the constants of any other system may be computed easily when those of any one system are known.

Of the five systems of expressing the constants, each has its own especial adaptability to a particular method of analysis. That the angle changes and Strassner's constants are especially adapted to Suter's and Strassner's methods and that Ruppel's constants are especially computed for the graphical method of fixed or conjugate points have already been indicated. Elsewhere[45] the writer has used the angle changes in developing the general form of the three-moment equation, resulting in an exceedingly simple formula as follows (see Fig. 24):

$$\beta_1 M_A + (\alpha_{R1} + \alpha_{L2}) M_B + \beta_2 M_C = -\alpha^\circ_{R1} - \alpha^\circ_{L2} \ldots\ldots (175)$$

The same constants have also been used by the writer in developing the generalized graphical analysis of restrained and continuous beams[46] and the slope-deflection equations[47], as follows:

$$M_{LR} = \frac{1}{\alpha_L\,\alpha_R - \beta^2}\left[\alpha_R\,\theta_L + \beta\theta_R - (\alpha_R + \beta)\,\frac{d}{l} \mp (\alpha_R\,\alpha^\circ_L - \beta\alpha^\circ_R)\right] . (176)$$

and,

$$M_{RL} = \frac{1}{\alpha_L\,\alpha_R - \beta^2}\left[\alpha_L\,\theta_R + \beta\theta_L - (\alpha_L + \beta)\,\frac{d}{l} \pm (\alpha_L\,\alpha^\circ_R - \beta\alpha^\circ_L)\right] . (177)$$

The constants of moment areas have been used by A. W. Earl[48], M. Am. Soc. C. E., in writing the slope-deflection equations; by Thomas F. Hickerson[49], M. Am. Soc. C. E., in developing his method of analysis; and by Ralph W.

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[44] "Cross' Constants for Members with Varying Moment of Inertia", by Fang-Yin Tsai and Tseng-I Yang, *Journal*, Tsing Hua Civ. Eng. Soc., Peiping, China, No. 2, July, 1931 (in English).

[45] "Theorem of Three-Moment in General Form", by Fang-Yin Tsai, The Science Repts., National Tsing Hua Univ., Peiping, China, Series A, Vol. I, pp. 19–36, April, 1923 (in English).

[46] "Generalized Graphical Analysis of Restrained and Continuous Beams", by Fang-Yin Tsai, *Journal*, Tsing Hua Civ. Eng. Soc., Peiping, China, No. 3, May, 1932 (in English).

[47] "Slope-Deflection Equations for the Analysis of Rigid Frames with Varying Moment of Inertia", by Fang-Yin Tsai, The Science Repts., National Tsing Hua Univ., Peiping, China, Series A, Vol. II, pp. 75–81, July, 1933. Similar equations have also been obtained by L. T. Evans, Assoc. M. Am. Soc. C. E., *Journal*, Am. Concrete Inst., October, 1931, p. 109; see, also, "Handbook of Rigid Frame Analysis", by L. T. Evans, Edwards Brothers, Inc., Ann Arbor, Mich., 1934.

[48] *Transactions*, Am. Soc. C. E., Vol. 96 (1932), p. 114.

[49] "Structural Frameworks", by T. F. Hickerson, The Univ. of North Carolina Press, Chapel Hill, N. C., 1934.

Stewart[50], M. Am. Soc. C. E., in his paper entitled "Analysis of Continuous Structures by Traversing the Elastic Curves." Besides their imperative use in the method of moment distribution, Professor Cross' constants have been

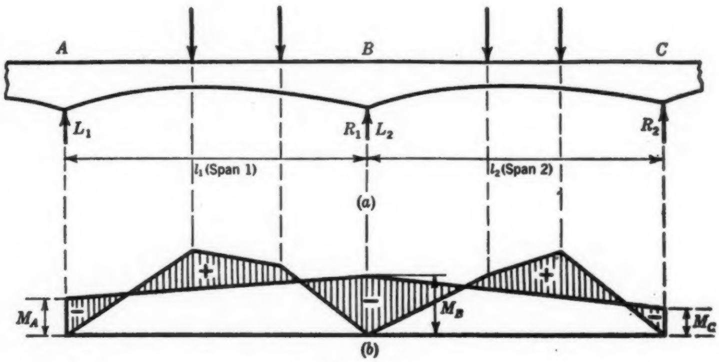

FIG. 24.

used by E. B. Russell[51] in his method of restraining stiffness, and may be also used in writing the slope-deflection equations, as follows:

$$M_{LR} = S_L \left[\theta_L + C_{LR}\,\theta_R - (1 + C_{LR})\,\frac{d}{l}\right] \mp M_{FL} \ldots\ldots (178)$$

and,

$$M_{RL} = S_R \left[\theta_R + C_{RL}\,\theta_L - (1 + C_{RL})\,\frac{d}{l}\right] \pm M_{FR} \ldots\ldots (179)$$

The comparison of the authors' F-constants with those of the five systems previously mentioned are presented in Table 4, in which the value of A is computed by the authors' Equation (7). This factor, $A + 1$, is introduced because, for reference of computation, the writer and others have all used the minimum moment of inertia, I', or I_c, whereas the authors have used the maximum moment of inertia, I_A, which is equal to $I_c\ (A + 1)$, or $I'\ (A + 1)$. Tables 3 and 4 not only facilitate the application of the authors' formulas to any method of analysis, but also furnish a comprehensive view of the relation between all the constants of the various systems.

Of the aforementioned systems, that of angle changes is perhaps the most fundamental and elegant inasmuch as such changes not only possess a definite physical meaning, but they also represent, directly, the predominant deformations of continuous structures, on the basis of which most of the methods of analysis are developed. Moreover, their use in certain methods, gives exceed-

[50] *Proceedings*, Am. Soc. C. E., October, 1934, p. 1125.

[51] "Analysis of Continuous Frames by the Method of Restraining Stiffness", by E. B. Russell, San Francisco, Second Edition 1934. Tables for the constants, $\frac{S_L}{4\,E\,I'}$, $\frac{S_R}{4\,E\,I'}$, C_{LR}, C_{RL}, $\frac{M_{FL}}{W\,l}$, and $\frac{M_{FR}}{W\,l}$, computed from Ruppel's tables may also be found in this book, pp. 24–42.

TABLE 4.—RELATIONS OF THE AUTHORS' F-CONSTANTS WITH THOSE OF THE OTHER SYSTEMS
(Units are All Ratios)

The authors' constants	Angle change	Moment area	Ruppel	Strassner	Cross
F_1	$\frac{E I' (A+1)}{l} \beta$	$\frac{I' (A+1)}{l} A_R v$ or $\frac{I' (A+1)}{l} A_L u$	$q v (A+1)$ or $p u (A+1)$	$\frac{A+1}{6} \phi_\beta$	$\frac{E I' (A+1)}{l} \frac{C_{RL}}{S_L (1 - C_{LR} C_{RL})}$ or $\frac{E I' (A+1)}{l} \frac{C_{LR}}{S_R (1 - C_{LR} C_{RL})}$
F_2	$\frac{E I' (A+1)}{l} \alpha_R$	$\frac{I' (A+1)}{l} A_R (1 - v)$	$q (1 - v) (A+1)$	$\frac{A+1}{3} \phi \alpha_R$	$\frac{E I' (A+1)}{l} \frac{1}{S_R (1 - C_{LR} C_{RL})}$
F_3	$\frac{E I' (A+1)}{l} \alpha_L$	$\frac{I' (A+1)}{l} A_L (1 - u)$	$p (1 - u) (A+1)$	$\frac{A+1}{3} \phi \alpha_L$	$\frac{E I' (A+1)}{l} \frac{1}{S_L (1 - C_{LR} C_{RL})}$
$F_4, F_6, F_8, F_{11}, F_{13}$	$\frac{E I' (A+1)}{W l^2} \alpha^\circ{}_R$	$\frac{I' (A+1)}{W l^2} A_0 g$	$t q (A+1)$	$\frac{A+1}{K 6} \phi_\beta \phi_t$	$\frac{E I' (A+1)}{W l^2} \frac{M_{FR} + C_{LR} M_{FL}}{S_R (1 - C_{LR} C_{RL})}$
$F_5, F_7, F_9, F_{12}, F_{14}$	$\frac{E I' (A+1)}{W l^2} \alpha^\circ{}_L$	$\frac{I' (A+1)}{W l^2} A_0 (1 - g)$	$s p (A+1)$	$\frac{A+1}{K 6} \phi_\beta \phi_e$	$\frac{E I' (A+1)}{W l^2} \frac{M_{FL} + C_{RL} M_{FR}}{S_L (1 - C_{LR} C_{RL})}$

ingly simple results, as in the case of the theorem of three moments presented by Equation (175). From Table 4, it is seen that the authors' F-constants are similar and proportional to those angle changes, and the former may be considered as the coefficients for the latter. Angle changes are, perhaps, most easily computed by the method of moment areas, applied either graphically, as shown by Ernst Suter[52], or by numerical summation, as shown by G. E. Large[53], Assoc. M. Am. Soc. C. E.

The authors' method is inherently approximate, since it requires the replacement of the actual I-curve by a substitute I-curve, and the latter can rarely fit the former exactly, especially when the actual I-curve is very irregular, or has some abrupt changes, as illustrated by the authors' Fig. 1. Although close approximation is not objectionable in the analysis of continuous structures (particularly in the case of reinforced concrete, in view of the many other uncertain and even untrue assumptions involved), the application of the authors' formulas for computing the F-constants requires much more labor than the moment-area method. It is regrettable that they have not given a numerical example to indicate how approximate the results of their method are and how much labor their computations require. For this purpose, the writer has solved the numerical example given by Professor Large[53] (Mr. Tsun-Kuai Liu has computed the various constants). Using the results of Professor Large's computations, the following constants of the moment-area method are derived (see Fig. 25): $A_0 = 89.15 \times 3 = 267.45$ kip-ft^{-2}; $A_L = 9.12 \times \dfrac{3}{10} = 2.736$ ft^{-3}; $v = 1 - \dfrac{19.62}{30} = 0.346$; $g = \dfrac{18.03}{30} = 0.601$; and $u = \dfrac{13.92}{30} = 0.464$. (The reason for using the multiplier, 3, in the computation for A_L is explained by Professor Large. The denominator, 10, is necessary since Professor Large used an end moment of 10 instead of unity, which is the value used by the writer in defining A_L and A_R). By Equation (167), $A_R = \dfrac{2.736 \times 0.464}{0.346} = 3.669$ ft^{-3}; or, using Professor Large's results, $A_R = 12.27 \times \dfrac{3}{10} = 3.681$ ft^{-3}. Both values of A_R are in close agreement. However, in order to check all the results exactly, the value, 3.669, will be used for A_R in all the computations. For the purpose of illustrating and comparing all the beam and load constants of the various systems, they have been computed for the same example in accordance with Table 3 and are arranged for comparison in Table 5. All the results in this table are cross-checked by computing them with 6-place logarithm tables to a precision of at least four significant figures, which, of course, is rarely necessary in ordinary practice.

[52] "Die Methode der Festpunkte", pp. 77–80.

[53] *Bulletin No. 66*, Ohio State Univ., Columbus, Ohio, November, 1932, p. 14. A similar but slightly different treatment by the same author is also found in *Transactions*, Am. Soc. C. E., Vol. 96 (1932), p. 102.

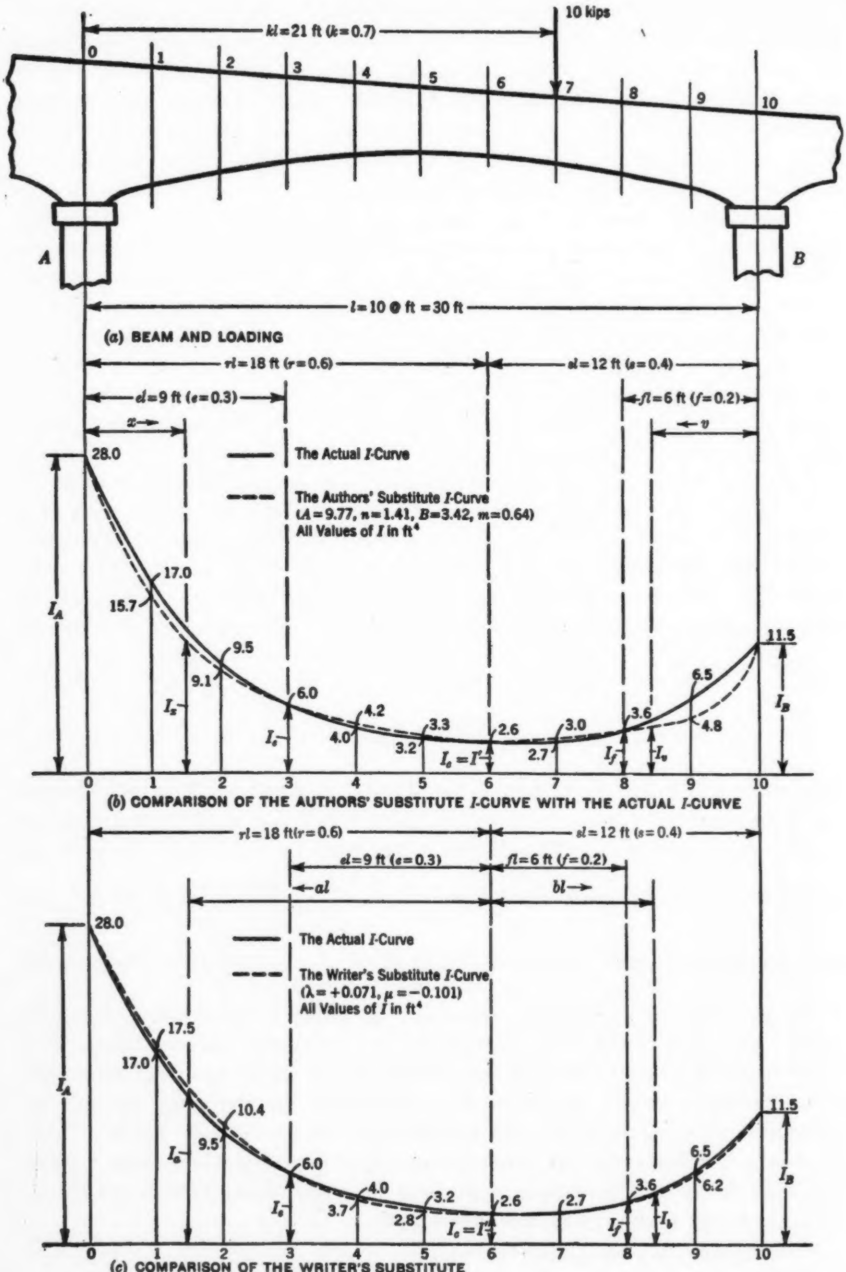
10 kips
kl = 21 ft (k = 0.7)
A
B
l = 10 @ ft = 30 ft
(a) BEAM AND LOADING
rl = 18 ft (r = 0.6)
sl = 12 ft (s = 0.4)
el = 9 ft (e = 0.3)
fl = 6 ft (f = 0.2)
The Actual I-Curve
The Authors' Substitute I-Curve
(A = 9.77, n = 1.41, B = 3.42, m = 0.64)
All Values of I in ft⁴
(b) COMPARISON OF THE AUTHORS' SUBSTITUTE I-CURVE WITH THE ACTUAL I-CURVE
al
bl
The Actual I-Curve
The Writer's Substitute I-Curve
(λ = +0.071, μ = −0.101)
All Values of I in ft⁴
(c) COMPARISON OF THE WRITER'S SUBSTITUTE I-CURVE WITH THE ACTUAL I-CURVE

FIG. 25.

Using the authors' method and referring to Fig. 25(*b*): By Equation (7),

$$A = \frac{28 - 2.6}{2.6} = 9.77\text{; by Equation (8), } n = \frac{\log \dfrac{28 - 6}{9.77 \times 6}}{\log \dfrac{0.3}{0.6}} = 1.41\text{; by}$$

Equation (10), $B = \dfrac{11.5 \times 2.6}{2.6} = 3.42$; and, by Equation (11),

$$m = \frac{\log \dfrac{11.5 - 3.6}{3.42 \times 3.6}}{\log \dfrac{0.2}{0.4}} = 0.64.$$

TABLE 5.—The Values of All Constants of the Various Systems for the Beam Shown in Fig. 25(*a*)

Angle changes	Moment areas	Ruppel	Strassner	Cross
$\alpha_L = \dfrac{1.470}{E} \dfrac{\text{radians}}{\text{kip-ft}}$ $\alpha_R = \dfrac{2.400}{E} \dfrac{\text{radians}}{\text{kip-ft}}$ $\beta = \dfrac{1.270}{E} \dfrac{\text{radians}}{\text{kip-ft}}$ $\alpha^\circ_L = \dfrac{106.71}{E}$ radians $\alpha^\circ_R = \dfrac{160\ 74}{E}$ radians	$A_L = 2.736$ ft^{-3} $A_R = 3.669$ ft^{-3} $u = 0.464$ $v = 0.346$ $A_0 = 267.45$ kip-ft^{-2} $g = 0.601$ Check: $A_L u = A_R v = 1.270$	$p = 0.2371$ $q = 0.3180$ $u = 0.464$ $v = 0.346$ $s = 0.1300$ $t = 0.1457$ Check: $pu = qv = 0.110$	$\phi_{\alpha_L} = 0.3812$ $\phi_{\alpha_R} = 0.6237$ $\phi_\beta = 0.6604$ $\phi_s = 0.2801$ $\phi_t = 0.1457$	$C_{LR} = 0.5291$ $C_{RL} = 0.8657$ $S_L = 1.2581\ E$ kip-ft $S_R = 0.7689\ E$ kip-ft $M_{FL} = 27.26$ kip-ft $M_{FR} = 52.55$ kip-ft Check: $C_{LR}S_L = C_{RL}S_R = 0.665\ E$ Modified: $S'_L = 0.6819\ E$ kip-ft $S'_R = 0.4167\ E$ kip-ft $M'_{FL} = 72.75$ kip-ft $M'_{FR} = 66.97$ kip-ft Check: $C_{LR}S'_L = C_{RL}S'_R = 0.361\ E$

With the foregoing four constants known, the authors' *F*-constants and Professor Cross' constants are computed by their "exact" and approximate formulas. The same constants are also computed from Professor Large's results in accordance with Tables 3 and 4. All the values are arranged in Table 6, in which the percentages of the errors in the values of the authors' formulas in comparison with those by the moment-area method are also shown. Since the results of Professor Large's computations are sufficiently accurate (he has checked them, at least partly, by models), the authors' "exact" formulas yield errors as great as 12% and their approximate formulas yield errors as great as 50% even for this quite ordinary case. As far as the writer's experience is concerned, the application of the authors' lengthy formulas involving fractional exponents requires considerably more time and is also more susceptible to errors than the moment-area method applied either graphically or by numerical summation. In view of the foregoing comparison, it is very doubtful whether the authors' method will replace the moment-area method and find favor among practising engineers. In fact, since the design of continuous structures usually requires a long cut-and-try process (as stated by the authors under the heading "Approximate Solution"), it seems

to the writer that there is absolutely no reason why, ordinarily, a designer should not design his tapered members in accordance with those prevailing types (straight, parabolic, and sharply curved tapers), for which many sets of very comprehensive tables and charts are now available, in order to save the enormous amount of labor which would be required if he chooses to do otherwise.

TABLE 6.—COMPARISON OF THE VALUES OF CONSTANTS COMPUTED BY MOMENT AREAS AND BY THE AUTHORS' METHOD
($A = 9.77$; $B = 3.42$; $n = 1.41$; and $m = 0.64$.)

Description	Moment-area method	THE AUTHORS' METHOD					
		"Exact"	Equation No.	Percentage error	Approximate	Equation No.	Percentage error
The Author's Constants:							
F_1	1.185	1.162	(63)	− 1.2	0.965	(112)	−18.6
F_2	2.240	2.355	(64)	+ 5.1	1.552	(113)	−30.7
F_3	1.369	1.358	(65)	− 0.8	1.253	(114)	− 8.3
F_6	0.500	0.483	(68)	− 3.4	0.348	(117)	−30.4
F^7	0.332	0.318	(69)	− 4.2	0.275	(118)	−17.2
The Cross' Constants:							
C_{LR}, or C_{AB}	0.5291	0.4935	(52)	− 6.7	0.6219	(52)	+17.5
C_{RL}, or C_{BA}	0.8657	0.8554	(53)	− 1.2	0.7702	(53)	−11.0
$\frac{S_L}{4\ E}$, or S_A, in feet³	0.3145	0.2972	(61)	− 5.5	0.3576	(61)	+13.7
$\frac{S_R}{4\ E}$, or S_B, in feet³	0.1922	0.1715	(62)	−10.8	0.2888	(62)	+50.2
M_{FL}, or M_A, in kip-feet	27.26	30.49	(42)	+11.8	26.80	(42)	− 1.7
M_{FR}, or M_B, in kip-feet	52.55	46.53	(43)	−11.5	50.56	(43)	− 1.9

From Fig. 25(b) it is seen that the authors' substitute I-curve falls short quite considerably in comparison with the actual I-curve at Section 9. Since a change in the I-value near the end of a member usually has a greater effect on the various functions than a change near the center, the large errors in the fixed-end moments computed by the authors' "exact" formulas may be attributed to this cause. The errors might possibly diminish somewhat if the substitute I-curve for the left half of the beam had been made to pass through the actual I-curve at Section 9 (in which case, $f = 0.1$; $I_f = 6.5$; and $m = 1.075$) instead of Section 8. However, if this is done, the substitute I-values to the right of Section 9 would be much too large, and the fitting of the substitute I-curve would also require a few trials. For this reason, the writer has tried to find some other equation for the substitute I-curve which will fit the actual I-curve better than that of the authors', and obtained the following results (see Fig. 25(c)):

$$I_a = I_C + (I_A - I_C)\left[\lambda\left(\frac{a}{r}\right)^2 + (1 - \lambda)\left(\frac{a}{r}\right)^3\right] \quad \text{.....(180)}$$

$$\lambda = \frac{\frac{I_e - I_C}{I_A - I_C}\left(\frac{r}{e}\right)^3 - 1}{\frac{r}{e} - 1} \quad \text{..................(181)}$$

$$I_b = I_c + (I_B - I_c)\left[\mu\left(\frac{b}{s}\right)^2 + (1-\mu)\left(\frac{b}{s}\right)^3\right] \quad \ldots\ldots(182)$$

and,

$$\mu = \frac{\dfrac{I_f - I_C}{I_B - I_C}\left(\dfrac{s}{f}\right)^3 - 1}{\dfrac{s}{f} - 1} \quad \ldots\ldots\ldots\ldots\ldots\ldots(183)$$

Equations (180) to (183) correspond, respectively, to the authors' Equations (6), (8), (9), and (11), with the difference that the variable abscissa ratios, a and b, are measured from the ordinate of I_C to the left and right, respectively, and that the values of λ and μ may be either positive or negative within certain limits. Applying Equations (181) and (183) to the foregoing example:

$$\lambda = \frac{\dfrac{6 - 2.6}{28 - 2.6}\left(\dfrac{0.6}{0.3}\right)^3 - 1}{\dfrac{0.6}{0.3} - 1} = +\ 0.071$$

and,

$$\mu = \frac{\dfrac{3.6 - 2.6}{11.5 - 2.6}\left(\dfrac{0.4}{0.2}\right)^3 - 1}{\dfrac{0.4}{0.2} - 1} = -\ 0.101.$$

The writer's substitute I-curve, together with the actual I-curve, is plotted in Fig. 25(c), from which it is seen that the fit is much better than that shown in Fig. 25(b), particularly at the left half where the curves almost coincide. The writer hopes that the authors will investigate the possibility of using the foregoing equations in their method.

The position of the working line shown in the authors' Fig. 12 is very reasonable and is in agreement with that used by German writers[35, 36]. However, a study should also be made of the condition at the intersections of beams and columns, which affects greatly their I-values and, consequently, the moments[54]. As indicated in Fig. 12, the authors have assumed the condition of Fig. 26(b) for considering the I-values of beams, which is also in agreement with German writers[35, 36]; but the condition shown in Fig. 26(c) is also very reasonable and may represent the actual condition more or less, particularly when the width of columns is considerable. For considering the I-values of columns, the conditions shown in Fig. 26(d) and Fig. 26(e) have usually been assumed, in which case the following angle changes may be easily derived when the cross-section of the columns is constant:

$$\alpha_R = \frac{(h')^3}{3\,E\,I\,h^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(184)$$

[54] "Analysis of Continuous Frames by the Method of Restraining Stiffness", by E. B. Russell, San Francisco, Second Edition, 1904, p. 45.

$$\alpha_L = \frac{h^3 - (h'')^3}{3\,E\,I\,h^2} \qquad (185)$$

and,

$$\beta = \frac{(h')^2\,(h + 2\,h'')}{3\,E\,I\,h^2} \qquad (186)$$

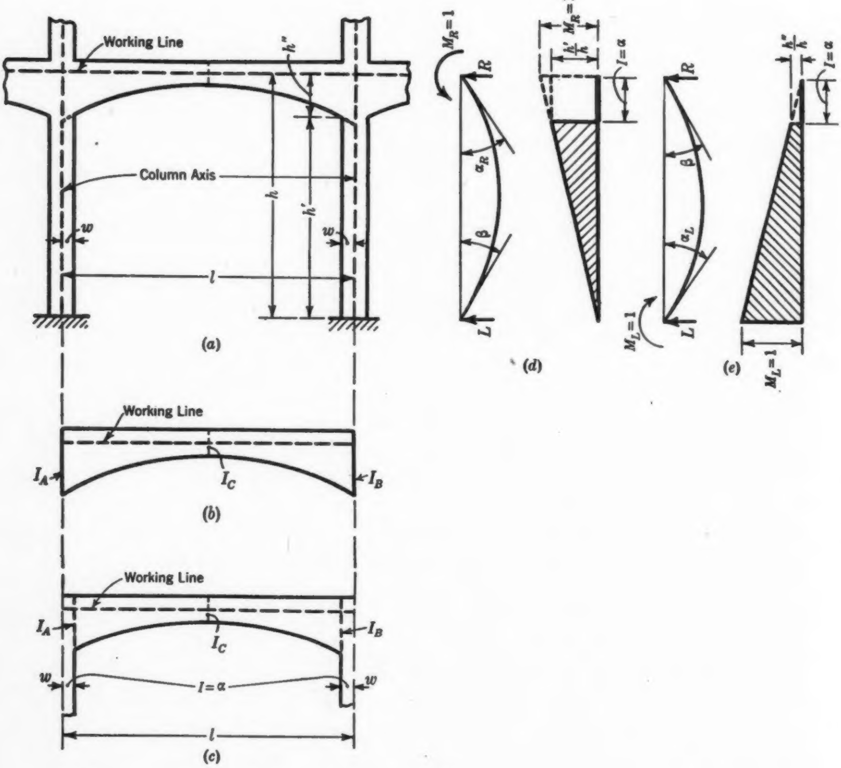

Fig. 26.

With the foregoing value known, Professor Cross' stiffness and carry-over factors may readily be computed. To make sure of the assumed conditions regarding the I-values and the working lines, experimental investigations with models in this direction will be of great value.

Its title indicates that the paper should cover a much wider scope than it actually does, inasmuch as the term, "structural members", as commonly understood, includes such members under the action of flexure, compression, tension, or torsion, or any combination of them. Actually, the paper treats such members under the action of flexure only, and, therefore, it would be more appropriate and also more precise if the word, "structural", in the title were changed to read "flexural."

A. A. Eremin,[55] Assoc. M. Am. Soc. C. E. (by letter).[55a]—An interesting method of computing the elastic coefficients of tapered members is presented in this paper. With the equations developed by the authors, a complete analytical analysis of perfectly rigid frames can be made.

The computations of the coefficients, F, may be simplified, as follows: A general integral expression of the elastic coefficients of tapered members is:

$$R = \int_0^l \frac{x^p\,dx}{I_x} \qquad (187)$$

In which p is an exponent of x that varies from zero to 5. In the most common cases (see Equation (24)), $p = 5$.

Substitute Equation (1) in Equation (187) as follows:

$$R = \frac{1}{I_0} \int_0^l \left[1 + A\left(\frac{x}{l}\right)^n\right] x^p\,dx \qquad (188)$$

Integrating and simplifying, Equation (188) becomes:

$$\int_0^l \frac{x^p\,dx}{I_x} = \frac{1}{I_0}\left[\frac{x^{p+1}}{p+1} + \frac{A}{l^n}\,\frac{x^{n+p+1}}{(n+p+1)}\right]$$
$$= \frac{l^{p+1}}{I_0}\left[\frac{1}{p+1} + \frac{A}{(n+p+1)}\right] \qquad (189)$$

The integrals entering Equation (189) may be expressed as:

$$\int_0^l \frac{dx}{I_x} = \frac{l\,R_0}{I_0} \qquad (190)$$

$$\int_0^l \frac{x\,dx}{I_x} = \frac{l^2\,R_1}{I_0} \qquad (191)$$

and,

$$\int_0^l \frac{x^2\,dx}{I_x} = \frac{l^3\,R_2}{I_0} \qquad (192)$$

From Equation (189), the values of R are:

$$R_0 = 1 + \frac{A}{n+1} \qquad (193)$$

$$R_1 = \frac{1}{2} + \frac{A}{n+2} \qquad (194)$$

and,

$$R_2 = \frac{1}{3} + \frac{A}{n+3} \qquad (195)$$

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[55] Assoc. Bridge Designing Engr., Div. of Highways, State Dept. of Public Works, Sacramento, Calif.

[55a] Received by the Secretary February 7, 1936.

Therefore, from Equations (33), (34), (35), (190), (191), and (192):

$$\int_0^l \frac{x(l-x)\,dx}{I_x} = \frac{l^3}{I_A}(R_1 - R_2) \quad \ldots\ldots\ldots\ldots (196)$$

$$\int_0^l \frac{x^2\,dx}{I_x} = \frac{l^3}{I_A} R_2 \quad \ldots\ldots\ldots\ldots (197)$$

and,

$$\int_0^l \frac{(l-x)^2\,dx}{I_x} = \frac{l^3}{I_A}(R_0 - 2\,R_1 + R_2) \quad \ldots\ldots\ldots\ldots (198)$$

Equation (189) is especially convenient in computing the loading coefficients, with bending moments in which the power of x is the highest, as in Equation (24).

Computation of the coefficients for tapered members may be further simplified by means of Strassner's tables[56], in which the elastic properties of members with variable sections are expressed by formulas for moment of inertia similar to those in this paper.

Assume that $N = \frac{I_0}{I_e}$; and that $t = \frac{x}{l}$. Then Equation (1) for the moment of inertia of a tapered member becomes the same as that introduced by Professor Strassner[57],

$$I_x = \frac{I_0}{1-(1-N)\,t^n} \quad \ldots\ldots\ldots\ldots (199)$$

The formulas developed by Messrs. Weiskopf and Pickworth suggest the construction of various tables and curves to simplify the computation of stresses in statically indeterminate frames. The authors have made a valuable contribution to the profession.

AUSTIN H. REEVES,[57] ASSOC. M. AM. SOC. C. E. (by letter).[57a]—The great worth of this paper is proved more conclusively with each increment of study. Having designed many types of structural frameworks containing members of variable moment of inertia by both conjugate points (fixed points) and the method of fixed-end moment distribution, it was with a critical (if not, antagonistic) state of mind that the writer approached the paper. This condition was caused by a casual glance disclosing plenty of calculus.

However, a start was made by checking about twenty of the equations against several designs which the writer knew to be correct. The accuracy of the results of the analytical treatment was thus established. Nevertheless, a further and much more exhaustive study was made, which produced an ever-increasing respect for the authors' treatment of the subject. For example, the writer solved the problem shown in Fig. 27, a beam with a lateral width of 12 in. Within 5 min it was determined that n should be less than 1.3.

[56] "Der Durchlaufende Rahmen", by A. Strassner.

[57] Newark, N. J.

[57a] Received by the Secretary April 20, 1936.

Within an additional 5 min it was decided that an accurate enough value of n would be 1.1; that is, since $e\ l = 4.9$ ft; $I_e = 4\,411$ in.^{4}; $I_A = 46\,656$ in.^{4}; and $I_C = 1\,728$ in.^{4}, A is found by Equation (7) to be 26. Then, it is a matter of simple substitution in Equation (8) to determine that $n = 1.1$.

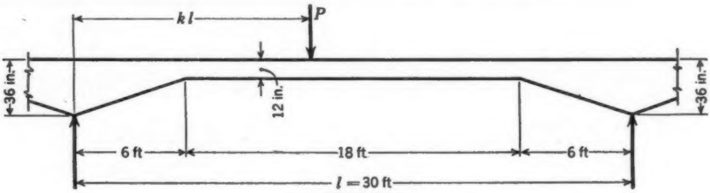

FIG. 27.

It is impossible to have more than one correct value for n. A correct substitute I-curve cannot run through any arbitrarily selected point because it must satisfy the following two conditions[58]:

(*a*) The area of the actual I-curve must be maintained in selecting a substitute curve; and,

(*b*) The distance from the left support to the center of gravity of this area must be the same for the substitute I-curve as for the actual curve because it controls the size of the carry-over factor, whereas the area controls the rigidity and is also a factor in determining the amount of the fixed-end moments.

It is incorrect to run a substitute I-curve, successively, through a number of points selected arbitrarily by measurements from the left support. Of course, such procedure would give to n a different value for each point selected.

The paper is so excellent that the writer hesitates to make suggestions. However, emphasis should be laid upon the fact that $e\ l$ and I_e should be selected correctly first (using Conditions (*a*) and (*b*) as criteria), and then the correct value of n can be computed by Equation (8) of the paper.

In conclusion, the writer wishes to express his admiration for the thorough manner in which so complicated a subject has been presented. The paper should be used extensively in the future as its full import becomes appreciated.

[58] *Transactions*, Am. Soc. C. E., Vol. 90 (1927), p. 90.

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DISCUSSIONS

INFLUENCE OF DIVERSION ON THE MISSISSIPPI AND ATCHAFALAYA RIVERS

Discussion

BY E. W. LANE, M. AM. SOC. C. E.

E. W. LANE,[18] M. AM. SOC. C. E. (by letter).[18a]—Briefly stated the author's contentions are:

(1) That the discharge capacity of the Mississippi River below Red River Landing has deteriorated considerably during the past fifty odd years.

(2) That this deterioration has resulted from a deposit of silt in this stretch which has reduced the channel area.

(3) That no corresponding deterioration has occurred in the channel capacity above Red River Landing.

(4) That the deposit below Red River Landing has occurred due to the diversion of water from the Mississippi at all stages through Old River, in conformity with the hypothesis of Guglielmini.

(5) That the slope of the Atchafalaya River has decreased in agreement with this same hypothesis.

The writer's discussion will cover these five points in order.

If, as the author contends, a reduction has occurred in the discharge capacity of the Mississippi River below Red River Landing in the 50 yr following 1882, of about 7 ft for a discharge of 1 100 000 sec-ft (which is equivalent to a reduction in discharge of about 250 000 cu ft per sec), it is a matter of major importance in connection with the flood control of the Mississippi River. Should this continue in the future, continuous construction will be necessary to prevent a progressive decrease in the degree of flood protection given by any system of works which does not remove the cause.

The method used by the author to prove this contention is sound, although so far as the writer knows it has not heretofore been used extensively, if at all.

NOTE.—The paper by E. F. Salisbury, M. Am. Soc. C. E., was published in November, 1935, *Proceedings*. Discussion on the paper has appeared in *Proceedings*, as follows: April, 1936, by Leo M. Odom, Assoc. M. Am. Soc. C. E.

[18] Prof. of Hydr. Eng., State Univ. of Iowa, Iowa City, Iowa.

[18a] Received by the Secretary March 18, 1936.

In order to check his results the writer has investigated the effect in a similar manner for the same discharge, and for discharges other than those used by the author. All current meter measurements, available to the writer, between 1 150 000 and 750 000 cu ft per sec have been used. The stages for each discharge within 50 000 of the even 100 000 cu ft per sec were adjusted to the nearest even 100 000 by using the discharge increment corresponding to that discharge, in a manner similar to that used by the author. All measurements that might have been affected by crevasses have been eliminated, except in 1882. Since the stages used in this year are so little, if any, above the bank level, the effect of crevasses on discharge would be negligible.

The results of these studies are shown on Fig. 3 in which the adjusted stages are plotted against their respective years for each of the four discharges. The solid line joins the points indicating the mean of the adjusted stages in each year. The lower dotted line joins points showing the lowest adjusted stage and the upper dotted line joins the points giving the highest adjusted stage. Above each set of points is given the number of observations on which the points were based. For each discharge a mean line has been drawn showing the general trend.

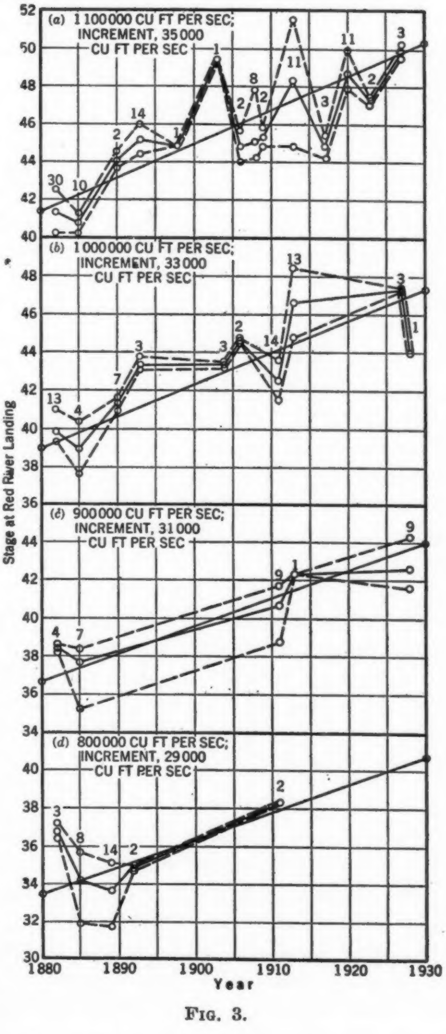

Fig. 3.

In each case the trend is unmistakably upward, the slope magnitude increasing somewhat with the discharge. This increase of stage for a given discharge is also indicated by plotting the discharge measurements near Red River Landing against stage, which is just another way of presenting the same data. A similar indication may be noticed in the Carrollton gaugings.

In Fig. 4 are plotted discharge rating curves for the years 1880 and 1930—based upon points taken from the mean lines of Fig. 3 at those years—for

the four discharges investigated. These curves have been extended by dotted lines to zero discharge at Gulf level (Elevation − 3.57 on the gauge) to indicate approximately the difference at other discharges. A comparison of these rating curves indicates very strongly that a reduction of the discharge capacity has occurred. The curves indicate a decrease of discharge capacity in 50 yr, at a 40-ft stage, of about 250 000 cu ft per sec.

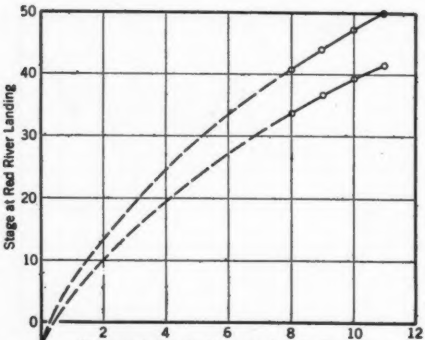

FIG. 4.—DISCHARGE RATING CURVES FOR 1880 TO 1930.

The evidence presented by the author that the maximum stages reached during the years since 1867 were tending to increase also is evidence pointing to deterioration, but is not so definite as the evidence based upon discharge, since the maximum stage reached in high-water years depended upon the strength of the levees, as the author states under "Conclusions." However, since the great flood years are few in number, the effect of the increasing strength of levees would probably be small. The value of 7.3 ft found by this method is not exactly comparable to the 6.9 ft found by the hydraulic method, since, assuming a uniform rate of deterioration, the 7.3-ft rise would occur between the middle of the first and the middle of the last period for which the stages were averaged, or 56 yr, whereas the period covered by the discharge method was 50 yr. The rise for 50 yr instead of 56 yr, assuming a uniform variation, would be 6.5 ft.

The author's second contention is that the deterioration of channel capacity below Red River Landing is due to a reduction of the channel area caused by a deposit of silt. A general form of equation for the discharge capacity of a river is:

$$Q = C\,A\,R^m\,S^n \qquad (1)$$

A reduction in capacity can result from a decrease in one or more of the elements, C, A, R, or S. For the Mississippi River below Red River Landing it can scarcely be due to a decrease in the slope, S, since the river discharges at a constant elevation into the Gulf of Mexico, and, for a given stage at Red River Landing, the average slope through this section of the river must be constant, both the fall and the length remaining the same. Small local or temporary changes of slope occur, but these changes tend to re-adjust themselves since the average slope cannot change. It seems improbable that a change in capacity could have resulted from a change in the value of C, the roughness factor. Except for the negligible increase in roughness, which no doubt occurred as a result of the construction of wharves and other works of Man, no reason is apparent to cause a permanent change of roughness in this section of the river, and so far as known, no one has ever suggested that one has taken place. Temporary changes of roughness, described by the author

as changes of "efficiency" however, do occur. The writer believes that these changes are to a greater extent due to changes in roughness than to deposits of silt, since the silt load of the river is insufficient to cause enough deposit to produce the decreases in "efficiency" as rapidly as they occur. It is probable that the change of roughness is due to the formation and obliteration of sand waves, which can take place in a short time. These changes of roughness, however, are temporary and tend to equalize over a period of years. A large part of the space between the upper and lower dotted lines in Fig. 3, and some of the variation of the average line, is no doubt due to this cause, although inaccuracies of measurement also form a part of this difference.

It is possible that a decrease in discharge capacity could occur as a result of a change in R without any change in A. This could result from a material widening and shallowing of the river without a change in area. With levees close to the banks, as they are in this stretch of the river, however, no such change could escape notice, as it would endanger many miles of levee. Although a minor change in width may have occurred, since no general widening has taken place, it seems safe to conclude that no appreciable deterioration has resulted from a change of the hydraulic radius except as it may have occurred as a result of a decrease in the channel area. Since the other possibilities seem to be eliminated, it is very probable that any deterioration that has occurred in the channel capacity below Red River Landing has been caused by the deposit of silt in the river bed, causing a decrease in the area, A, and a corresponding decrease in the hydraulic radius, R.

The stage-discharge curves of Fig. 4 indicate that the increase in stage is different for each discharge. This shows that the amount of the filling of the bed cannot be determined by taking the difference in the stages at the ends of the period for any given discharge since the result obtained would depend upon the discharge selected. It should be possible, from the dimensions of the channel, and the slope at the beginning of the period, to compute the amount of the filling (assuming it to be distributed in proportion to the distance from the Gulf) which would cause the deterioration indicated, but it cannot be determined from the stage differences alone.

The third contention—that no material deterioration has occurred above Old River—seems well established, since it is supported not only by the discharge measurements but also by the measurement of great numbers of cross-sections, which show a small increase in area. Because of the variation that may occur at any one cross-section in the river (due to the passage of a sand wave or from a number of other causes) reasoning regarding the action of the river bed based upon a few measurements at one cross-section, or in a stretch of river, while useful as evidence, is not very conclusive. For this reason too much weight should not be given the data presented by the author showing the loss of cross-section at Red River Landing during the high waters of 1929 and 1932; nor to the measurements of the areas at the two ends of the Pointe a la Hache diversion. However, when arguments are based upon hundreds of cross-section measurements scattered over a stretch of river

and, to a lesser extent, on many measurements at a given locality spread over a long period of time, they should carry considerable weight.

The fourth contention (that the deposit below Old River was caused by the diversion of water from the Mississippi at all stages through Old River, in conformity with the hypothesis of Guglielmini), is likely to provoke considerable discussion. Probably no engineering controversy has continued so long and resulted so inconclusively as that dealing with the Guglielmini hypothesis and its application to the Mississippi River. As early as 1816 this hypothesis was used as an argument against spillways for flood control. It was strenuously combatted by Humphreys and Abbot, as a result of their studies, and strongly supported by Eads, who applied it with great success to the deepening of the South Pass at the mouth of the Mississippi by jetties in the years 1875 to 1879. Largely as a result of this success, the hypothesis was adopted by the Mississippi River Commission as a basic principle; but as a practical means of flood control it proved nearly a complete failure. Nevertheless, it exerted a strong influence on the policy of the Mississippi River Commission until the 1927 flood and was a considerable obstacle in the path of flood control by spillways until it was forced into the background by the strength of the sentiment for spillways which developed as a result of that flood. With the author's paper it has come to the fore again. It is greatly to be hoped that the truth or fallacy and the bearing of it may be established, as the uncertainty has been a serious obstacle in the way of establishing a sound flood-control policy for this great river.

Personally, the writer is now inclined to accept the Guglielmini hypothesis, although at one time he was very skeptical of it because of the difficulty (which has since been removed) of reconciling the apparently conflicting evidence when applying it to the Mississippi River. Of course, the wording of the Seventeenth Century does not fit in with the modern more exact conceptions, but it should be remembered that no such accurate conceptions existed at that time. Moreover, part of the inexact expression may have been caused by inapt translation. Perhaps the greatest argument against the hypothesis, as far as the Mississippi River is concerned, is its apparent failure when applied by the Mississippi River Commission to flood protection. By plotting a mean relation of the silt load carried by the Mississippi and applying this relation to the discharge of past floods as they would have occurred had no levees been built, and again to these same floods under conditions of confinement actually produced by levees, it can be shown that the small enlargement which has taken place in the river is all that the Guglielmini hypothesis would indicate.

The error of the members of the original Mississippi River Commission, therefore, was not in believing in the Guglielmini hypothesis, but in not using the data available to them and applying the hypothesis quantitatively to their problem. It is believed that an application of the foregoing method to the conditions in the relatively short stretch at the mouth of the river covered by jetties (where the action went on for 365 days per yr) would show that it should succeed in that case when it failed in the case of flood control. A similar application to the reduction of flow in the Mississippi

below Red River Landing, due to the greater discharge down the Atchafalaya, would throw valuable light on the action of the river in that stretch.

Another argument against the Guglielmini hypothesis is that used by Humphreys and Abbot based on the fact that they found no exact relation between the discharge and the silt load of the river. Recent studies[19] have shown why this is not necessarily a disproof of the theory. The writer believes that when applied quantitatively, with due regard to the time element involved, the Guglielmini hypothesis will be found to be a sound general relation as far as it goes, although of course many other factors may enter to prevent its working out in special cases. The hypothesis is incomplete, in that it does not consider the variation of the solids load as a factor, and to make accurate prediction this factor also must be added.

Independent, however, of the soundness of the Guglielmini hypothesis, if silting is progressing below Red River Landing and not above that point, it is logical to expect the cause of this silting to be something operating below that point and not above it. One of the strong arguments in favor of the applicability of the Guglielmini hypothesis to this case would seem to be the difficulty of explaining the apparent decrease of capacity below Old River without utilizing it.

Diversion through Old River, however, is not the only phenomenon that might cause the deposition below that point, according to the hypothesis of Guglielmini. Before the removal of the raft from the Atchafalaya River, probably the greater part of the flow of the Red River reached the Mississippi; now only a very small part of it reaches the river. This diversion to the Atchafalaya of the greater part of the flow of the Red River may have a greater effect on the Mississippi channel below Red River Landing than the flow that passes out of the Mississippi through Old River. Judging from the size of the trees said to be growing around these rafts and the damage that followed the removal of the raft, it is reasonable to suppose that the flow down the Atchafalaya had been small for many years, and, therefore, that the greater part of the Red River flow had passed down the Mississippi. In a long period of time the Mississippi would have adjusted itself to this flow and when it was practically discontinued as a result of the enlargement of the Atchafalaya, the effect would be much the same as if that quantity of water had been abstracted from the Mississippi through Old River. In considering the action of the Guglielmini hypothesis on the Mississippi below Red River Landing, therefore, it is necessary to consider not only the actual diversion of water from the river through Old River, but also the diversion of that part of the Red River flow which formerly entered the Mississippi, but now passes down the Atchafalaya. In this problem the character of the solids load of the Red River is a factor and would have to be considered to arrive at accurate quantitative results.

A silting up of the river might take place during a series of dry years, but this would be expected to occur along the length of the stream not just in one part. However, the bottom of the river in the section below Old

[19] *Engineering News-Record*, Vol. 115, October 17, 1935, p. 538.

River is considerably below sea level and the flow section there would not decrease with decreased flow as rapidly as in sections of the river farther up stream. This would give rise to relatively lower velocities below Old River in low-flow periods, and hence more tendency to silt. Whatever the cause of the decreasing capacity, it seems to have been acting fairly continuously throughout the past fifty years, and if it is due to a long-continued drought it would be necessary to have longer rainfall records than are available to prove it. Some light might be cast on this question by tree-ring studies if they covered the entire drainage area of the river.

The author's proof of his fifth contention, that the slope of the Atchafalaya River has decreased in agreement with the Guglielmini hypothesis, is not very complete. He has shown that the Atchafalaya has enlarged materially in cross-section, that the bank elevations along the stream have become higher, and he has given data tending to prove that a discharge of 300 000 cu ft per sec occurred at substantially the same stage from 1890 to date. Since the area was increased and the discharge remained the same, the velocity has been reduced, and he concluded, therefore, that the slope had been reduced enough to cause this decreased velocity. As previously stated, the general discharge relation is $V = C\ A\ R^m\ S^n$. If V remains the same, the increase in area (and presumably the consequent increase in R) can be offset by a decrease in either, or both, C or S. Although it seems improbable that the value of C has changed in the Mississippi it is by no means improbable in a river scouring as actively as the Atchafalaya was during the period under consideration. The irregular cutting of the banks and bed which occurred under such circumstances might easily cause an increase in roughness. The great turbulence of the river is very noticeable at high flows, and the computed roughness factors show considerably higher values than an ordinary river channel.

The upper section of the Atchafalaya River has a comparatively steep slope and the lower section, which is composed largely of shallow lakes, has a very flat slope. The material scoured from the upper end as the river enlarged, and a considerable part of the silt load brought into the upper end of the river, has been deposited in the upper end of the shallow lakes and has formed a delta. This condition has caused a rise in the water level at this point and, to this extent, a decrease in the slope of the Atchafalaya. The rise in level has been strikingly shown by the tendency of the water about 1926 to flow to the Mississippi through the Plaquemine Lock, whereas, formerly, it always tended to flow in the opposite direction. It is probable, therefore, that there is some decrease in slope of the upper end of the Atchafalaya, due to its tendency to form a delta at the head of the lakes; but it seems doubtful whether this would be enough to offset the increase in both area and hydraulic radius which occurred as a result of the enlargement of the river bed. The late John Augustus Ockerson, Past-President, Am. Soc. C. E., has shown[20] that the enlargement of the river was due largely to the concentration of velocities that occurred as a result of the steep slopes where the water spread

[20] *Transactions*, Am. Soc. C. E., Vol. LVIII (1907), p. 1.

out at the end of the leveed section of the river, and the progressive moving down of this high-velocity section as the levees were extended. It is probable that the decrease of slope which offsets the increased area is due largely to the extension of the confinement of the river by levees, which occurred during the period of river enlargement. Immediately preceding Table 11 the author states that the increase in gauge height at Melville between 1892 and 1907 is due to levee confinement. An increase of stage at Melville, the level at the upper end of the river remaining the same, would result in a flatter slope.

On the whole the author has presented a strong argument to support the contention that the discharge capacity of the Mississippi below Old River is decreasing. As a result of his studies of Mississippi River hydraulics, however, the writer has learned that different, apparently equally good, lines of proof sometimes lead to directly opposite conclusions and, therefore, every available method should be used before a final conclusion is formed. It would seem that the best way to prove or to disprove a contention that the stretch of the Mississippi below Old River had been filling would be to make another cross-section survey similar to that made from 1893 to 1898. The writer could find no published record of such a survey since that date. This method and any other possible method of indicating the deterioration, should be investigated so that its existence can be established or disproved as conclusively as possible. Other possible investigations that might throw light on this subject are a study of the frequency of the various stages below Old River throughout the period of record and a comparison of the area below a certain datum in the cross-sections shown by the discharge measurements at Red River Landing. A comparison of stage relations below Red River Landing, however, would probably not indicate the deterioration as the stages at all points are likely to be changed proportionally. In this section of the river, also, a comparison of low-water stages would not indicate deterioration because of the great depth of the water, even at the lowest flows.

Assuming that the deterioration is proved conclusively, it would seem that serious thought should be given to Mr. Eads' proposal to place a dam across the upper end of the Atchafalaya and force the ordinary flow of the Red River down the Mississippi. The dam should be provided with gates which could be opened during floods to make available the discharge capacity of the river. A lock would have to be provided at this dam, and perhaps another lock and dam farther down stream would be required to take care of navigation.

If the deterioration is due to the working out of the Guglielmini hypothesis, any increase in the diversion down the Atchafalaya River at low or medium flows would increase the rate of deterioration and thus offset a part, or all, of the benefit from the increased diversion. Of course, it would be possible to increase the Atchafalaya capacity fast enough to keep ahead of the deterioration, but in the present state of the knowledge of such action it would be a race with an uncertain outcome.

The Guglielmini hypothesis has been used so often as an argument against flood control by spillways that an argument for the hypothesis might errone-

ously be considered to be an argument against spillways. The writer believes that much of the argument against flood-relief spillways based on the Guglielmini hypothesis has not considered the entire situation. When the first flood-control works were built on the Mississippi, the river had carved for itself a channel of a size which was a result of the action of a great variety of conditions in the past decades and probably did not change materially over a period of years. The construction of levees tended to confine the water and has caused an enlargement of the river channel. This enlargement is so much smaller than the proponents of the confinement theory expected that it has not been seriously considered, but it is all that the theory indicates should take place. A flood-relief spillway will cause some deposit in the river down stream from it, but here, again, the amount will probably be much smaller than the opponents of spillways anticipate. Since these spillways will operate only for short periods with considerable intervals between (during which time the channel will often carry increased flows due to the confinement by the levees of all the floods which do not bring the spillways into action) it is probable that the increased scouring caused by the levees will take care of the deposit caused by the spillway. Of course, the river below the spillway will not enlarge as rapidly as it would if no spillway was present, and the flow continued to be confined between levees; nor would it enlarge as fast as the river up stream from it, but there would probably not be a decrease in section. That the result would be a net enlargement is indicated by the increase in section area that has occurred in the river above Red River Landing in spite of the frequent occurrences of crevasses (which have the same effect as spillways).

The three effects, enlargement due to confinement by levees, deposit below spillways, due to diversion through them, and deposit due to constant diversion or subtraction of tributaries, can be considered to act largely independently of each other. Whether the river fills or enlarges depends on the relative magnitude of the effects. Above Old River the enlargement due to confinement has been greater than the deposit resulting from crevasses. Below Old River the effect of the diversion and subtraction of tributaries, plus the deposits caused by crevasses, seems to have been greater than the enlargement due to confinement.

At present, it is not possible to give quantitative answers to what the effects of these various changes being made on the river will be, but with the rapid advance of knowledge of solids transportation in the last few years, and with the collection of a reasonable amount of data on the characteristics of the bed and suspended load of the Mississippi and its principal tributaries, it should soon be possible to make definite quantitative predictions based on adequate data and sound reasoning.

AMERICAN SOCIETY OF CIVIL ENGINEERS

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DISCUSSIONS

STABLE CHANNELS IN ERODIBLE MATERIAL

Discussion

By Messrs. R. E. Ballester, and Gerald Lacey

R. E. Ballester,[10] Esq. (by letter).[10a]—Table 1 of the paper shows a wide variation in values of C and n for Equation (1) applied to non-silting and non-scouring velocities. The formula proposed by the writer (Curve No. 10, in Fig. 1 and Table 1), namely, $V_0 = 1.01\ d^{0.44}$, in English units, has a lower exponent, n, than the other formulas. The writer wishes to present some additional data, that has confirmed its application to the channel sections in which it was observed (Rio Negro, Argentina).

In a stretch of 10 km (6.2 miles) of the main channel of the irrigation system, of 2.00-m (6.6-ft) depth, 32.75-m (107.4-ft) bottom width, and 45 cu m per sec (1 589 cu ft per sec) of discharge, it was observed in 1926 that the mean velocity was not sufficient to prevent silting and the formula, $V_0 = 1.01\ d^{0.44}$ (English), or, in metric units,

$$V_0 = 0.52\ d^{0.44} \qquad (19)$$

was then graphically established[20], taking this fact into consideration.

The channel silted gradually and, in 1933, the writer advised that the velocity in the channel be increased by widening the notched fall openings at the end of a 10-km stretch, that was causing some back-water. The advice was followed in 1934 and in 1936, after two years, the silting action has disappeared in the last 2 km ($1\frac{1}{4}$ miles) of channel. The mean velocity arrived at is somewhat higher than would be necessary, because some scour has occurred along the bottom of the channel, the slopes being unaffected. The velocities registered before and after the widening of the fall are shown in Fig. 8,

Note.—The paper by E. W. Lane, M. Am. Soc. C. E., was published in November, 1935, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: February, 1936, by R. C. Johnson, M. Am. Soc. C. E.; and April, 1936, by Messrs. E. S. Lindley, J. C. Stevens, C. R. Pettis, Harry F. Blaney, and Sigurd Eliassen.

[10] Prof. of Applied Hydraulics, Univ. of Buenos Aires, Buenos Aires, Argentine Republic.

[10a] Received by the Secretary March 23, 1936.

[20] "Velocidades y Coeficientes de Aspereza para el Cálculo de Canales en el Rio Negro", Contribuciones al Estudio de Las Ciencias Físicas y Matemáticas, Série Tecnica, Vol. III, Pt. 5, p. 435, Buenos Aires, 1927.

together with the Kennedy and Rio Negro formulas. If the Kennedy formula had been used, the writer is certain that the channel would have scoured seriously.

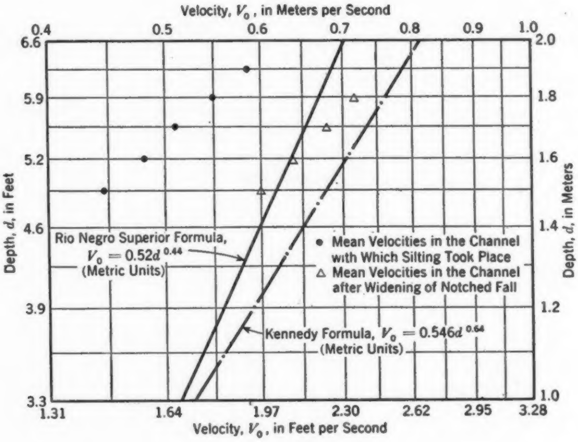

FIG. 8.

The channel cross-section now shows a tendency to scour along the bottom and to assume a semi-elliptical form. The bed-width-depth ratio of 16 for this cross-section is too high for this particular location. In alluvial valleys the soil is far from homogeneous, and there are great changes in the quality of soil, which are difficult to establish from the general topographical survey that precedes the design and alignment of the channel.

Regarding the composition of soil in the channel of the Rio Negro System, the data in Table 6 may be of interest for comparison with other channels.

TABLE 6.—ANALYSIS OF SILT AND BOTTOM SOIL IN THE MAIN CHANNEL OF THE RIO NEGRO SYSTEM

Sample No.	Station, in kilometers	Depth	Clay contents (percentage)	Sieve Analysis		
				Percentage retained on Sieve No. 100	Percentage retained on Sieve No. 200	Percentage passing Sieve No. 200
1........	18	0.20 m below water	19.4	10.0	23.0	67.0*
2........	16	0.20 m below water	22.2	3.0	7.0	90.0*
3........	12	Bottom of channel	4.6	14.2	49.1	36.7†
4........	212.6	Bottom of channel	2.0	37.7	38.0	24.2†

* Samples of silt deposited on the side slopes.
† Samples of soil forming the channel bed (not silt).

Of course, there are many variables that combine to stabilize the shape of a channel and some of them do not depend on soil condition. In the Rio Negro System there are favorable conditions tending to minimize the troubles due to silting. Irrigation delivery is suspended in winter from the end of May to the middle of August (eighty days on the average), just when the river is in flood, carrying its heavier silt load. At the end of spring (November), there are some small floods, but the intakes are ordered

closed during the peak of the flood (not more than a day), because the trouble caused by delaying the delivery of water to the 130 000 acres in the System is less than would be caused by a continuous silting of the distribution system. Special conditions such as these, which are peculiar to each system, may explain the wide variation in the critical velocity formulas in Fig. 1 and Table 1 of the paper.

GERALD LACEY,[21] ESQ. (by letter).[21a]—Congratulations are due the author on his very able discussion of the various theories of silt transport extant when his paper was prepared. Engineers in India will immediately be struck by the very different conditions obtaining in Colorado and the great alluvial plains of India. The problem of regime flow is, in general, much simpler in India than on the Colorado River. Kennedy postulated for his channels that they should be flowing in their own self-silted beds and within self-silted side slopes; effectively they were regarded as flowing in an unlimited alluvial plain of the same silt grade as that transported. Recently, the writer has developed, on the same lines, "a general theory of flow applicable to channels flowing uniformly in incoherent alluvium"[22]. If a heavily charged channel is excavated in a medium other than the silt transported, or if the channel is afforded no opportunity of forming a reasonably thick boundary of the material transported, other factors will come into play. In certain circumstances the problem may merge from Gilbert's stream traction into flume traction[23].

In India canal cross-sections flowing within a boundary of silt deposit present an undeniable cup-shaped, curved, cross-section closely approximating a semi-ellipse. Professor Lane has quoted Kennedy's observations that channels carrying a heavy load had practically vertical sides and horizontal bottoms. This condition is prevalent where the channels are in "cut" and the wetted perimeter is less than the discharge can demand, or both. All recent observations have shown that when the channel is in "fill" (that is, formed by artificially constructed embankment) and the wetted perimeter assigned is sufficiently large, the cross-section tends to become semi-elliptical. Engineering practice favors the excavation of channels with horizontal bottoms, and for this reason many large channels with bottoms of stiff clay, or material more tenacious than the silt transported, preserve their horizontal beds to some extent. It may be easier for a limited depth of bed silt to slip and slide over a tenacious sub-stratum, than for the tenacious material to be picked up in the center. The difference between a scouring velocity and a silting velocity may be great. In certain embanked channels in India, with a bottom above ground level no attempt, from motives of economy, was made to raise the bed itself by "filling." It was anticipated that the defect would be corrected in time by silt deposits. It is these channels that now present the characteristic semi-elliptical cross-section. The Kheri Branch of the Sarda

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[21a] Received by the Secretary February 20, 1936.

[22] "Uniform Flow in Alluvial Rivers and Canals", by Gerald Lacey, *Minutes of Proceedings*, Inst. C. E., Paper No. 4893.

[23] "The Transportation of Débris by Running Water", by G. K. Gilbert, *Professional Paper No. 86*, U. S. Geological Survey.

Canal in its lower reaches was excavated in fine sand that was almost incoherent. These reaches now present a curved cross-section in many instances of great regularity.

Professor Lane draws attention to the fact that on the Imperial Valley canals the non-silting velocity is considerably greater than the writer's formulas would indicate. The fine heavy silt charge demands a velocity of the same order as that demanded by a smaller charge of coarse silt in India. Equation (8) quoted by Professor Lane can apply only in the case of a regime charge. It was advanced by the writer as "a very rough qualitative formula for the diameter in inches of the predominant type of silt transported" and was intended mainly to assist in computing the critical velocities of natural streams in sand, shingle, or boulders, rather than as a basis of design. The silt factor is proportional to $\frac{V^2}{R}$, and it is this ratio which is required before channels can be designed. Mere measurement of the size of the particle without observations of the local velocity required to transport the charge is of little assistance. The silt charge in the Colorado River appears to be of great importance and abnormal as compared with conditions in India. The writer also feels that there are certain elements of flume traction that cannot be entirely dissociated from channel behavior on some, at least, of the Colorado canals. In channels with flume traction there is no limit to the surcharge that can be forced down provided slope is available and the banks possess some tenacity. In rivers flowing in deep alluvium, the surcharge is thrown down, and the river moves to a flank and picks up a fresh charge which it sweeps forward.

The remark by Professor Lane that the capacity of a stream to transport silt in suspension is probably proportional to its "turbulence" and to the energy expended, is illuminating. According to Gilbert the prime mode of transportation is saltation rather than suspension, and the difference is important. With flume traction and a heavy surcharge the turbulence necessary to drive the boundary layer forward would also occasion an abnormal suspended charge.

In the writer's view "turbulence" is susceptible of definition, and the ratio, $\frac{V^2}{R}$, for any grade of silt and irrespective of the charge, epitomizes "turbulence." It is thus found that fine silt with a heavy surcharge may well demand the same value of $\frac{V^2}{R}$ as a coarse silt with a smaller charge in another locality. Again, the ratio, $\frac{V^2}{R}$, is intimately connected with the energy concept.

Professor Lane has given a comprehensive list of factors entering into channel behavior, but the writer feels that it is preferable to write them down as dimensioned variables. The Buckingham theory[24] summarizes very con-

[24] "On Physically Similar Systems; Illustration of Use of Dimensional Equations", by E. Buckingham, *Physical Review*, Vol. IV, Series 2, pp. 345–376, 1914; also, "Model Experiments and the Form of Empirical Equations", by E. Buckingham, *Transactions*, A. S. M. E., Vol. 37, pp. 263–296.

veniently modern practice in dimensional analysis. Briefly, the method consists of writing down all the known independent dimensioned variables, and determining dimensionless arguments therefrom. These arguments are then correlated by ordinary statistical methods. Results obtained in such a manner have the merit of dimensional homogeneity. Mere dimensional homogeneity, however, does not denote accuracy, and a poor degree of correlation may reveal that other important variables remain to be sought.

If the number of independent variables is a, the number of dimensionless arguments is equal to $a-3$. Dealing with the independent variables historically, Chezy selected, effectively, four independent variables which left him with $a-3$ (equal to one) dimensionless arguments for which it remained only to ascertain the numerical value. Thus, $\frac{i\,R}{\rho\,V^2}$ = the Chezy number, in which, in addition to the notation of the paper, i = energy gradient; R = hydraulic radius; V = mean velocity; and ρ = density of water.

Experience showed that not only was the Chezy number a function of the roughness, but it was also a function of R and possibly, as Kutter thought, of S. An essential variable had been omitted. Attention is drawn to the manner in which the ratio, $\frac{V^2}{R}$, enters the Chezy number, a number that must survive with the addition of other dimensionless arguments as knowledge increases.

Osborne Reynolds added, as a fifth variable, the viscosity of water. This added another dimensionless argument, the classic Reynolds number; thus:

$$\frac{i\,R}{(\rho\,V^2)} = K\left(\frac{R\,V}{\nu}\right)^m \qquad \text{.................... (20)}$$

in which ν is the kinematic viscosity of water, and $\frac{R\,V}{\nu}$ is the dimensionless Reynolds number. It will be observed that the acceleration due to gravity, g, does not enter as an independent variable in Osborne Reynolds' relation. Osborne Reynolds' experiments were carried out with pipes. The boundaries were rigid and the shape was a constant.

In 1930, the writer suggested that the rugosity, or hydraulic roughness, of the regime alluvial stream bed was implicit in the slope and the hydraulic mean depth which they assumed. In open channels, g enters as an independent variable. With the addition of this sixth variable a third dimensionless argument is evolved. This is none other than the Froude number, and,

$$\frac{i\,R}{(\rho\,V^2)} = K'\left(\frac{R\,V}{\nu}\right)^m\left(\frac{V^2}{g\,R}\right)^p \qquad \text{.................. (21)}$$

Attention is again drawn to the manner in which the ratio, $\frac{V^2}{R}$, enters the Froude number. The silt factor is merely a simple proportion to the ratio, $\frac{V^2}{R}$, and, therefore, as long as dimensional analysis is used as an instrument

in hydraulic analysis the ratio, $\frac{V^2}{R}$, must persist as a fundamental element, and with it the silt factor, which more appropriately could have been termed "turbulence." In regime channels, "turbulence" with dimensions, $\frac{L}{T^2}$, would appear to be as real a factor as kinematic viscosity with dimensions, $\frac{L^2}{T}$ (in which T = time).

The writer[22] has shown how these dimensioned arguments can be fitted to empirical relationships defined by him. Thus, substituting for the energy gradient in terms of the geometrical slope, density, and the acceleration due to gravity,

$$\frac{S\,g\,R}{V^2} = K'' \left(\frac{R\,V}{\nu}\right)^{-\frac{1}{3}} \left(\frac{V^2}{g\,R}\right)^{\frac{2}{3}} \quad \text{................(22)}$$

If the silt factor is treated as being proportional to $\frac{V^2}{R}$, Equation (22) may be solved and will prove the basis of all the writer's formulas other than those involving the discharge, thus leaving the silt factor in the ratio form, $V \propto g^{\frac{5}{9}}\ \nu^{-\frac{1}{9}}\ R^{\frac{2}{3}}\ S^{\frac{1}{3}}$; and this relationship is true for any regime channel in incoherent alluvium whether fine sand, coarse sand, shingle, or boulders. The very low power of the kinematic viscosity should be noted and the complete disappearance of the "silt factor", which is inherent in the dimensions assumed by the channel.

The formula may be re-cast in the form, $V \propto g^{\frac{5}{9}} \nu^{-\frac{1}{9}} \sqrt{\frac{R}{V}}\ \sqrt{R\,S}$, from which it follows if an equation of the Chezy type is required, $V \propto g^{\frac{5}{9}}\ \nu^{-\frac{1}{9}}\ f^{-\frac{1}{4}}\ R^{\frac{3}{4}}\ S^{\frac{1}{2}}$. The Manning power of $\frac{2}{3}$ becomes $\frac{3}{4}$, and the rugosity coefficient, n, is replaced by $f^{\frac{1}{4}}$.

The introduction of the wetted perimeter, P, as a variable gives the solution, $\frac{P}{R} \propto \left(\frac{R\,V}{\nu}\right)^{\frac{1}{3}} \left(\frac{V^2}{g\,R}\right)^{\frac{1}{3}}$, from which $P \propto Q^{\frac{1}{2}}$; and $\frac{P}{R} \propto V$. All the most recent work in India tends to confirm the expression, $P \propto Q^{\frac{1}{2}}$, as fundamental.

Since, by the writer's Equation (5)[22], $P = 2.668\ Q^{\frac{1}{2}}$:

$$P^2 = 7.12\ P\ R\ V \quad \text{......................(23)}$$

and, dividing through by $P\,R$,

$$\frac{P}{R} = 7.12\ V \quad \text{.........................(24)}$$

This is the writer's fundamental shape formula, and it is most unfortunate that so many of the data presented in the past are incomplete in respect of the wetted perimeter. Thus, a diagram showing $\frac{P}{R}$ plotted against V

would be more illuminating than Professor Lane's Fig. 2, in which he has used the diameter of the silt as a basis for deriving the shape from the writer's formulas instead of the velocity. It is not clear from Professor Lane's paper whether or not the writer's shape formula is confirmed. The relation between the silt factor and the diameter clearly breaks down when there is a heavy silt charge, the fine silt behaving in the same manner as coarse silt in a normal channel. The writer would be very grateful if Professor Lane would plot the ratio, $\frac{P}{R}$, against V for all the Colorado River data.

Professor Lane states that the velocity along the bed should be sufficient to move all the material brought into the canal and yet not be so high as to cause the sub-grade of the canal to scour. He then goes on to state that the excess of velocity over that sufficient to move the transported material, which will attack the sub-grade, depends upon the material of the sub-grade. This concept clearly involves conditions approximating those of flume traction in which the transported material is pushed forward over a harder more coherent, and possibly coarser sub-grade. When these conditions obtain, a horizontal bed is readily understandable. The maximum allowable velocity along the banks depends not only upon their material but also largely upon the shape factor. The contours of equal velocity ("isovels") plotted in Fig. 3, proceeding from a central nucleus of maximum velocity, show very clearly what a channel would choose to be, as opposed to what Man makes it. The suggestion of the semi-ellipse is plainly discernible and also the cramping warping effect of the vertical sides. With a tough sub-grade and a heavy surcharge, nearly all the work on the perimeter is done on the bed and little on the sides. The bed would then be nearly horizontal and the sides vertical.

If a channel is too narrow according to Equation (24) it will "kick" at the sides and this is the main reason for bank erosion. Furthermore, if the wetted perimeter is correct, but the bed as constructed is horizontal, the "velocity nucleus" of the isovel diagram will shift about. With a channel properly constructed to a semi-elliptical trace the greatest velocity will be permanently located over the center of the channel. If a channel is constructed in material similar to that of the silt transported, or is a natural channel flowing in its own alluvium (for example, any sandy river), the central velocity will hammer the bed and curve it. Nature has nothing to say to horizontal beds when a river is flowing in a true alluvial plain.

The writer is of the opinion that a final solution of the problem of silt transport is to be found by investigation on the lines of dimensional analysis outlined by him. In India where the problem is simpler the writer's solution appears to fit a great mass of data. In the Colorado River, where the silt charge is excessive and there are many disturbing factors (notably the merging of stream traction into flume traction), practical design must depend largely on local conditions. The sub-grade as a factor is unknown in India, an unlimited bed of the same material as the silt transported being postulated. These are the conditions governing all great rivers in true alluvium. An examination of the Colorado Delta should prove interesting.

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DISCUSSIONS

TRUSS DEFLECTIONS: THE PANEL DEFLECTION METHOD

Discussion

BY MESSRS. A. W. FISCHER, AND L. E. GRINTER

A. W. FISCHER,[20] ESQ. (by letter).[20a]—In this paper the author has added another method for calculating the deflections of a truss by the analytical method, which is stated to solve "the problem of truss deflections in a simpler and more direct manner than other analytical methods." To the writer it seems that there are several analytical methods for trusses of a certain type, that are as simple as that of the author, which is based upon set formulas. One solution which is very simple for computing the deflections of Warren trusses is the "method of elastic weights"; another is the "geometric method"; and still another (which is a very flexible method for solving the vertical and horizontal deflections of any shape truss) is the method of "relative deflections"[21].

To determine the deflections of panel points in the author's Example 1, for instance, relative to Member 5–6, assume the truss supported at Panel Points 6 and 5 instead of at the left and right ends, being hinged at Point 6 and supported horizontally at Point 5.

Table 12 gives the values of u for the members for unit vertical load at various panel points. This table is not absolutely necessary, in the method of relative deflections, but is presented for a better understanding of certain values given in Table 13.

Using Example 1 and the elongations given by Mr. Shoemaker it is easy to enter the values as given in Table 13 from which the deflections can be computed. All the values given in Table 13 are readily determined

NOTE.—The paper by Louis H. Shoemaker, M. Am. Soc. C. E., was published in November, 1935, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: January, 1936, by Messrs. David B. Hall, and E. Mirabelli; February, 1936, by Messrs. William Bertwell, and Robert H. Hurlbutt; March, 1936, by Messrs. A. A. Eremin, T. P. Noe, Jr., David A. Molitor, and Glenn L. Enke; and April, 1936, by Fang-Yin Tsai, Assoc. M. Am. Soc. C. E.

[20] Care, Pennsylvania Sugar Co., Philadelphia, Pa.

[20a] Received by the Secretary March 11, 1936.

[21] "Stresses in Statically Indeterminate Structures", by Prof H. Yu, National Wuhan Univ., Wuchang, Hupeh, China, Second Edition, 1935, pp. 13 to 39.

TABLE 12.—VALUES OF u* FOR UNIT VERTICAL LOAD AT PANEL POINTS

Member	Unit Vertical Load at Panel Point		
	3	1	0
3-5	$+\frac{30}{36}\times 1$	$+\frac{30}{36}\times 2$	$+\frac{30}{36}\times 3$
3-6	$-\frac{46.8}{36}$	$-\frac{46.8}{36}$	$-\frac{46.8}{36}$
4-6	0	$-\frac{30}{36}\times 1$	$-\frac{30}{36}\times 2$
3-4	0	$+1-\frac{1}{7.2}$	$+1-\frac{1}{7.2}\times 2$
1-3	0	$+\frac{30}{35.52}\times 1$	$+\frac{30}{35.52}\times 2$
1-4	0	$-1.387+\frac{1}{5.191}$	$-1.387+\frac{1}{5.191}\times 2$
0-4	0	0	$-\frac{30}{31}$
0-1	0	0	$-\frac{43}{31}$

* The stresses in all members due to unit vertical loads at various panel points.

TABLE 13.—EVALUATION OF RELATIVE VERTICAL DEFLECTIONS OF PANEL POINTS

Member	$\frac{S L}{1\,000\,A}$	$u_m\left(\pm\frac{p}{r^*}\right)$	Moment center	$g_{mn}=\frac{S L}{1\,000\,A}u_m$	u_s	†$G_n=\Sigma g_{mn}$	ΣG_n	$\frac{S L}{1\,000\,A}u_s$	Total vertical deflection from Point 6, in inches, $\times\frac{30\,000}{12}$	Point	Total vertical deflection from Point O, in inches, $\times\frac{30\,000}{12}$	Total vertical deflection from Point O, in inches
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)
3-5	−220.8	$+\frac{30}{36}$	6	−184.0	......	......	......	......	....	6	2 367.7	0.95
3-6	+131.0	......	...	......	$-\frac{46.8}{30}$	−184.0	−184.0	−170.3	−354.3	3	2 013.4	0.81
3-4	+ 14.4	......	...	......	+1.0	......	......	+14.4	−339.9	4	2 027.8	0.81
3-4	+ 14.4	$-\frac{30}{216}$	(3)	−2.0	......	......	......	......	....	...	....	.....
4-6	+189.3	$-\frac{30}{36}$	3	−157.8	......	−159.8	......	......	....	...	....	.....
1-3	−222.5	$+\frac{30}{35.52}$	4	−188.0	......	......	......	......	....	...	....	.....
1-4	+210.0	$+\frac{30}{155.73}$	(4)	+ 40.5	−1.387	−147.5	−491.3	−291.3	−1 122.5	1	1 245.2	0.50
1-2	+159.3	......	...	......	+1.0	......	......	+159.3	(−963.2)	2	1 404.5	0.56
0-4	+369.0	$-\frac{30}{31}$	1	−357.2	......	......	......	......	....	...	....	.....
0-1	−286.0	......	(1)	......	$+\frac{43}{31}$	−357.2	−848.5	−396.7	−2 367.7	0	0.0	0.00

* r = the lever arm of Member m–n about its center of moments, expressed in feet.

† $G_n=\Sigma g_{mn}$ includes the summation of all the g_{mn}– values of all the members having a moment center at Panel Point n. g_{mn} represents the quantity, $\frac{S\ L}{A\ E}u_m$, for Member mn.

and once the method is mastered it is easy to calculate the vertical deflections of a truss. Furthermore, the method of relative deflections may be readily applied to a truss bridge with inclined top and bottom chords and with unequal panel lengths, *p*.

The paper is based on analytical methods, but for solving the secondary stresses in a truss by the slope-deflection method or by distributing fixed-end moments, the relative vertical and horizontal deflections are only required for the value of Δ for each member, Δ being the displacement, at right angles to the axis of the member of one end with respect to the other end, and to get these values of Δ the analytical method is not as flexible as the simple Williot diagram. In the author's closure it might be well for him to demonstrate the calculation of the horizontal deflections for his Example 1 and then show how he would calculate the values of Δ (that is, the displacement at right angles to the axis of the member, of one end with respect to the other end) and compare the time against that required by using a Williot diagram.

In conclusion, the writer will state that he sees no particular advantage in using the "panel deflection" method for the solution of certain types of trusses. There are certain other types in which the method is satisfactory, but as certain set formulas must be available for use, it loses its simplicity against other analytical methods.

L. E. GRINTER,[22] ASSOC. M. AM. SOC. C. E. (by letter).[22a]—The summation process of deflection computation has long been used in the study of truss deflections. The innovation proposed by Mr. Shoemaker is that the contributions of the several panels, rather than the individual contributions of the separate members be added to obtain the deflection. However, since the effect of each panel must be evaluated by considering the individual changes in lengths of the separate members, it is evident that the amount of computations involved in the two procedures must be about the same. If the method proposed by the author seems simpler than the procedure of summing, directly, the effects of the individual members, it is merely because the standard textbook treatment of the latter method (elastic or angle weights) is unnecessarily cumbersome. When the method of angle weights is reduced to its fundamental conceptions, it will be found to be essentially the same as the method suggested by Mr. Shoemaker.

There are two basic conceptions involved in the method of angle weights. First, the physical picture that deflection can be computed by summing the products of angle changes and lever arms; and, second, the conjugate-beam conception which cares for the effect of dissymmetry. The author has considered only symmetrical structures with symmetrical loading; and, accordingly, he has neglected consideration of the conjugate beam. Hence, his procedure for determining the deflection of a symmetrical truss is to find the upward deflection of the end relative to a fixed center-vertical member or relative to a fixed central-chord member. One notices at once the similarity to the

[22] Prof. of Structural Eng., Agri. and Mech. Coll. of Texas, College Station, Tex.

[22a] Received by the Secretary April 10, 1936.

use of the Williot diagram for those cases in which the use of the Mohr rotation diagram can be avoided.

When the author discusses the correspondence of his method with the method of computing beam deflections in the following passage, "to compute the deflection of a point of a beam of constant moment of inertia with reference to a tangent to the elastic curve of that beam, the sum is taken of the products of the angular change of every vertical section multiplied by its distance from the point", he is referring to relative and not to absolute deflections. If this method of computing relative deflections is to be used to compute the maximum deflection of a beam that is unsymmetrically loaded, the procedure illustrated by Fig. 22 becomes necessary. The deflection, Δ_1, is obtained as the statical moment of the $\frac{M}{EI}$-area from A to B about the point, B, where the deflection, Δ_1, exists. Then, the value of θ_1 is Δ_1 divided by L, the span of the beam. The point of maximum deflection, or D, is located such that the area of the $\frac{M}{EI}$-diagram from A to D is equal to θ_1. Then, finally, the maximum deflection Δ_2, is computed as Δ_3, the deflection of the beam at A from a tangent drawn to the elastic curve at D.

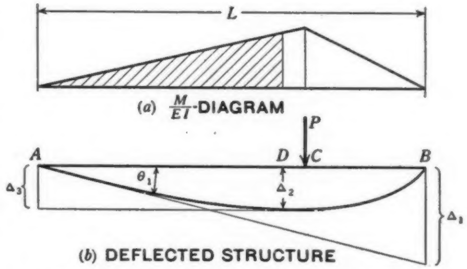

FIG. 22.—COMPUTATION OF THE MAXIMUM DEFLECTION.

The foregoing procedure, or a similar one, would be involved in the use of the author's method for the study of an unsymmetrical truss. Evidently, the difficulties involved would make the use of the method of panel deflections rather cumbersome for such trusses. The study of dissymmetry by the use of elastic weights (which involves the conjugate beam) would be much more satisfactory. The Williot-Mohr diagram is also a convenient method of determining the deflection of an unsymmetrical structure.

In the selection of a method for computing deflections, one should make a distinction between those methods that give a single deflection and those that give rise to the entire deflected load line. The author's procedure belongs to the second classification along with the method of elastic weights and the Williot-Mohr diagram. Castigliano's theorem and the method of virtual work give a single deflection for each separate computation. Hence, the determination of a single deflection or the calculation of a reaction (indeterminate structure) for fixed loads follows more simply by the use of these tools. On the other hand, a method that develops the entire elastic curve (deflected load line) is more satisfactory for the determination of an influence line as the shape of the deflected load line.

In summarizing, the writer feels that the panel-deflection method is an interesting and useful conception that in practice should be limited to the field of symmetry. It bears the same relation to the method of elastic weights

that the moment-area method (statical moments of moment areas) bears to the conjugate-beam method. One should not lose sight of the fact that the deflections obtained are relative to a fixed tangent. The panel-deflection method belongs to the classification of methods that develop the entire elastic line. No known procedure can compete with the method of virtual work for the computation of a single truss deflection. The author has performed a service in calling this interesting procedure to the attention of engineers and educators.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

SEDIMENTATION IN QUIESCENT AND TURBULENT BASINS

Discussion

BY MESSRS. HARRY H. HATCH, HARRY H. MOSELEY, AND GEORGE J. SCHROEPFER

HARRY H. HATCH,[10] M. AM. SOC. C. E. (by letter).[10a]—The author should be commended for his efforts to derive equations which will eliminate errors in the application of the Hazen formulas on sedimentation. Mr. Hazen applied the behavior of one particle size to an entire material in suspension. Obviously, this application can be justified only when the assumed value of t can satisfy the existing conditions.

In discussing Hazen's paper, "On Sedimentation", Robert Spurr Weston, M. Am. Soc. C. E.,[11] states:

> "Mr. Hazen bases a large number of propositions on one value of t. It would be interesting to substitute other values of t,—for instance, infinity. In the latter case, t would be a quantity vastly different from those mentioned in the paper."

Professor Slade has attempted to do this and more.

It would be interesting to know the difference, if any, between "time of settling", "time of sedimentation", and "time of subsidence", as used by the author. If they all mean the same, it would have been better for the sake of clearness to use only one of the expressions throughout the paper.

The plotting on logarithmic scale of observed data for Curve (2), Set 2, Fig. 8, did not result in a simple equation satisfying all the points. Had there been numerous observational data on the same material in suspension, an average condition could have been assumed and expressed easily. If it is absolutely essential to fit a curve that will pass through every observation point (or reasonably so), the graph on the logarithmic scale could be broken up into sections, and a simple equation for each section within the limits could be expressed. However, a logarithmic-probability scale plotting of the data of Curve (2), Set 2, Fig. 8, resulted in a flat and smooth curve passing prac-

NOTE.—The paper by J. J. Slade, Jr., Esq., was published in December, 1935, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: February, 1936, by Thomas R. Camp, M. Am. Soc. C. E.

[10] Engr. in Chg., Cobble Mountain Reservoir, Springfield Water-Works, Springfield, Mass.

[10a] Received by the Secretary February 4, 1936.

[11] *Transactions*, Am. Soc. C. E., Vol. LIII (1904), p. 77.

tically through all the observation values. The experimental value of 48% sludge in suspension at the end of 10 min, falls well on this curve; for which point the author had computed a percentage of 39.6 (see Table 1(*b*)), or a variation of 17½% from the observed value.

With respect to Fig. 9, the author states:

"* * * the distribution function cannot be found from the curve for turbulent sedimentation, so that wherever the writer has applied the general formula, Equation (31), it has been a matter of mere curve fitting."

He then begins with the formidable Equation (31) and reduces it "simply" to Equation (35), which is no less formidable; and then assumes certain unknown values and "by trial and error, or by any other method of curve fitting", arrives at Equation (36), which is cumbersome and far from being simple for practical work. The author further states,

"In a way, of course, this curve-fitting procedure furnishes a means of determining the distribution function (the *B*-values), but since the distribution function is not what one usually wants, no importance should be attached to it."

The question, therefore, is why do all that work?

The following equation was readily obtained by the writer from logarithmic plotting of the observation data given in Fig. 9:

$$Y = \frac{66}{t^{0.156}} \qquad (46)$$

in which Y = turbidity in parts per million, and t = time, in hours. Equation (46) will be near enough for all practical purposes. Its graph will vary slightly from that of Fig. 9 for values of t greater than 4 hr. There is no real justification, or no real objection to the path of the curve on Fig. 9 beyond the fourth hour, where it appears that turbidity does not decrease below 52 ppm, for practical purposes.

The writer really fails to discern any particular advantage of all the work done by the author to obtain Equation (36), when a simple equation answering the purpose could be obtained readily.

The settling time of a particle, practically speaking, depends on its velocity, and the velocity, in turn, depends upon the size of the particle. For the same sample material the sedimentation and gradation curves bear a definite relation. In fact, each depends on the other and one could be obtained from the other. No satisfactory general formula has been obtained for the distribution of particle sizes represented by a gradation curve, because of the complex nature of the material. It has been found more practical to make the necessary number of tests and plot their graphs. These test results are usually plotted on natural, semi-logarithmic, or logarithmic-probability scales. Often an error in observation can be detected in the plotted graph. Logarithmic-probability scale plottings result in a flatter curve and are better for the purpose.

It appears that, practically in all cases, without actual test runs, certain terms or factors of the equations proposed by the author can not be obtained. The author states the necessity of determining numerous constants experi-

mentally before making any practical application of the theories developed in the paper. The characteristics of material in suspension are so complex that general application of any set of constants is questionable. To this uncertainty must be added the tedious work involved and probability of errors in solving complicated equations. The writer questions whether direct tests or simpler methods, similar to gradation curves, would not be more practical in every respect. However, the paper affords a good exercise in mathematics and Equation (27) is an improvement over Equation (24), Hazen's second formula.

HARRY H. MOSELEY,[12] ASSOC. M. AM. SOC. C. E. (by letter).[12a]—Much has been written about the design of sedimentation tanks, but the correct design still remains an open question. Professor Slade's paper presents a mathematical approach to this problem. He has attempted to establish a relation between the settling of similar solids in a quiescent basin and in a turbulent basin. He has developed a theory which is dependent upon a group of constants and suggests three sets of constants that effect the solution of his formulas: "(1) Those which represent the characteristics of the sediment; (2) those which represent the characteristics of the settling basin; and (3) those which depend on both the character of the sediment and the degree of agitation of the fluid."

Much experimentation and work has been done and many data have been collected on the constants pertaining to the characteristics of the sediment. The point, it seems, may be raised regarding the density of the solids in the liquid. Was it the author's intention to include this in the characteristics of the sediment and thus affect the first constant by a certain degree? There is a large store of data available for the second set of constants. Any settling tank is a potential source. The information desired probably should include: The type of tank inlet and tank outlet; the length and the width of tanks; the depth or length-depth ratio; the type of sediment-removal equipment; the type of operation of removal equipment; the baffling linear velocity and direction of flow of liquid (average and bottom if possible); the direction of flow; the characteristics of solids in the liquid; and the results obtained by the tank. It is appreciated that many of these data may seem to be superfluous, but they all have a bearing on the results obtained from a basin. In many respects, the third set of constants, may be absorbed by the first two sets. However, in the design of grit chambers, or like structures, this set of constants would have an effect on the solution of the problem.

Much of the information pertaining to settling tanks in sewage treatment works has been obtained by the use of the Imhoff cone. Under which classification would the author place these data? Would it be quiescent, the same as a vertical cylinder, incomplete turbulence, or complete turbulence? These available data in many cases will have to be used in the determination of the second set of constants.

At the end of his paragraph discussing "Stratification of Sediment" Professor Slade states that "at a depth greater than $v_m\ a$ [see Fig. 6], the density

[12] Asst. Engr., George B. Gascoigne, M. Am. Soc. C. E., Cons. San. Engr., Cleveland, Ohio.

[12a] Received by the Secretary February 19, 1936.

will be what it was throughout the basin originally." This statement is based on the fact that the sediment in the basin has variable hydraulic subsidence values ranging from v_m to v_M. With a sediment of constant hydraulic subsidence characteristics it is possible after time, a, to have that part of the basin below $a\,v_m$ or $a\,v_M$ ($a\,v_m = a\,v_M$ when the hydraulic sibsidence characteristics are constant) of the same density as the original, throughout the basin. Theoretically, all particles of the same hydraulic subsidence characteristics are subsiding at the same rate; hence the absence of any flocculating action of the sediment. However, in a basin filled with sediment of variable hydraulic subsidence characteristics there is a flocculating action or the natural combination of suspended particles into larger aggregates. These larger aggregates also have hydraulic subsiding characteristics which are greater than the original particles and, consequently, settle more rapidly. Therefore, as soon as settling is started in this basin a portion of the liquid will have a density of suspended solids greater than that throughout the basin originally.

This flocculating action, of course, increases as the density of the suspended solids increases, even though the hydraulic subsidence characteristics of the original suspended solids in each case are the same before the settling starts, because there is a greater possibility of the suspended solids combining with each other as they start settling, thereby making larger aggregates. For example, a settling tank having a detention period of 1.5 hr will remove roughly 60% of the suspended solids from a raw sewage of 400 ppm, whereas it will remove approximately 45% from one containing only 100 ppm. It is true that in the raw sewage of 400 ppm, some of the solids will have greater hydraulic subsidence characteristics than in the one of 100 ppm; yet it is very doubtful whether this increase in hydraulic subsidence characteristics of some of the suspended solids is great enough to account for an increase of one-third in the efficiency of the tank. This increase can be accounted for better by the flocculating action of the suspended solids in the tank. It seems that the statement made at the end of the paragraph on "Stratification of Sediment", previously referred to, is true mathematically, but due to the flocculating action of the solids, the actual approaches it only as a limit.

The two examples presented by the author based on his interpretation of the original data substantiate his theory in a marked degree. However, it will require agreement in many actual field tests to prove his theory fully. It is hoped that Professor Slade's paper will further stimulate research in the field of sedimentation.

GEORGE J. SCHROEPFER,[13] JUN. AM. SOC. C. E. (by letter).[13a]—An admirable attempt to set up, by means of equations and formulas, some of the factors affecting sedimentation of solids, is contained in this paper. In view of the importance of sedimentation in water purification and sewage treatment, and in the separation and classification of ores and chemicals, all investigations that contribute to the fund of knowledge on this subject are distinctly to be appreciated.

[13] Asst. Chf. Engr., Minneapolis-St. Paul San. Dist., St. Paul, Minn.

[13a] Received by the Secretary March 30, 1936.

Although this discussion will be limited to the sedimentation of sewage solids, many of the principles of sewage sedimentation to which reference will be made, are equally applicable to the theory of sedimentation in general. The writer is inclined to favor the evaluation and determination of various engineering phenomena by means of formulas, but in the case of sedimentation, especially of sewage solids, the large number of influencing factors makes such a method of attack questionable when applied to an actual and practical situation.

Several times in the course of the paper the author makes reference to the application of his theory to sedimentation of sewage solids. Sewage is an extremely variable material, changing in quantity and quality, and in various characteristics which affect its settleability hourly, daily, and seasonally. The results of the application of a purely theoretical basis of analysis to a particular sewage under certain fixed or assumed conditions is valueless and even misleading when considered in terms of a varying material under varying external influences. A certain degree of control can be exercised over some of the variables, but in case of others it is necessary to accept the material as it exists, amenable to sedimentation, or otherwise.

In considering sewage sedimentation, the writer groups the various influencing factors under four headings, as affected by: (*a*) Characteristics of the liquid; (*b*) characteristics of the solids; (*c*) characteristics of the design; and, (*d*) miscellaneous effects.

(*a*).—The principal characteristics of the liquid which affect the settleability of solids are its specific gravity and viscosity. Of the two, it can be demonstrated that viscosity is the more important factor. With a liquid, such as water, the fluid characteristics, therefore, are closely affected by temperature. From a study of monthly average results at two large plants the writer has found a close correlation between the reduction accomplished and the sewage temperature. The reduction in suspended solids at the two plants closely approximates a 0.65% increase in reduction per degree Fahrenheit of temperature change; and in reduction of bio-chemical oxygen demand, 0.50% for every degree of temperature change. These results closely approximate those expected from purely theoretical considerations.

(*b*).—Under characteristics of the solids can be included such factors as size, shape, specific gravity, concentration of particles, natural flocculation, and artificial coagulation and coalescense. Sewage contains particles varying largely in size, shape, and specific gravity. In specific gravity alone particles vary from less then 1.0 to 2.65, or more. As an example of the importance of this factor as affecting a purely theoretical consideration, the hydraulic subsiding value of a particle 0.50 mm in size and having a specific gravity of 1.1, is 3.3 mm per sec; and 53.00 mm per sec for a particle of the same size having a specific gravity of 2.65.

Concentration of particles also plays an important part in the sedimentation of sewage solids, as evidenced by the fact that a study of the data collected on the effect of this factor indicates that, for a given detention period, a sewage containing 200 ppm of suspended solids will have its solids removal increased approximately 30% when compared with the sewage containing

50 ppm of suspended solids. Flocculation, coagulation, and coalescence exert a variable effect depending upon local conditions.

(*c*).—Under characteristics of the design can be included such factors as detention period, linear velocity of flow, depth and ratio of length and depth, inlet and outlet effects, shape of tank, baffling, and mechanism effects.

The first factor mentioned—namely, detention period—plays a very important part in the removal effected by a settling tank. The writer has collected data from more than forty settling-tank installations throughout the United States, which data, for those who desire to express the results of sedimentation data in the form of a equation, can be stated in the following formula:

$$R = C_1 - \frac{C_2}{t + C_3} \qquad (47)$$

in which R is the normal expectancy, in percentage removal of suspended solids; t is the detention period, in hours; and C_1, C_2, and C_3, are coefficients depending on the characteristics of the sewage and of the settling tanks under the particular conditions existing at that time. In view of the importance of the detention period provided, considerable thought should be given this factor in order to be assured that the desired period of detention is actually secured. Sewage treatment plants investigated indicate that the actual flowing-through time in the tanks may be as low as 10% of the theoretical detention period. This would indicate that, in some plants, a considerable expenditure of funds is, in effect, wasted by reason of inefficient design.

With respect to linear velocity of flow, a considerable variation in thought exists as to the desirable maximum velocities beyond which it is not advisable to go. Although this variation ranges from 4 to 59 mm per sec, the writer is inclined to believe that the linear velocity of flow for average flow conditions should preferably be not more than 2 ft per min (approximately 10 mm per sec), depending, however, on influencing conditions and requirements.

Inlet and outlet effects play a very important part in the length of tank actually effective for settling purposes and, in view of their importance, should be given considerable thought in design.

(*d*).—Under miscellaneous effects might be included such factors as currents caused by wind, eddies, and difference in temperature and biological activities. At one plant investigated the writer observed that with a wind velocity estimated at 25 miles per hr the velocity of the surface sewage in the 90-ft square tank was observed to be such as to cover the distance to the outlet in 2 min. The theoretical detention period in the tank was 1.5 hr. Methods effective in reducing this action consist of baffling, provision of free-boards, and the limitation of the size of tanks. When air and sewage temperatures are materially different, short-circuiting, mixing, and stratification may result.

Realizing the value of such investigations as the author reports, the writer has attempted to point out some of the factors which affect the evaluation of sedimentation data, but which, in the case of sewage solids subsidence, at least, are likely to make the methods of determination suggested by the author uncertain and possibly even misleading.

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DISCUSSIONS

SUCCESSIVE ELIMINATION OF UNKNOWNS IN THE SLOPE DEFLECTION METHOD

Discussion

BY MESSRS. FANG-YIN TSAI, A. FLORIS, A. W. FISCHER, AND L. E. GRINTER

FANG-YIN TSAI,[17] ASSOC. M. AM. SOC. C. E. (by letter).[17a]—By a simple process the method presented by Professor Wilbur can be adjusted to apply to the case of continuous structures with variable moments of inertia.

Slope-deflection equations for the case of variable moments of inertia have been presented by L. T. Evans[18], Jun. Am. Soc. C. E., and also by the writer[19], both results being almost identical. These equations involve certain coefficients known as angle changes which express the effect of variable moment of inertia, and the forms of these equations are rather cumbersome, and, therefore, inconvenient for use in this method. The writer has deduced the following equations, using the various constants of the method of moment distribution developed by Hardy Cross[20], M. Am. Soc. C. E.:

$$M_{ab} = S_a\,[\theta_a + C_{ab}\,\theta_b - (1 + C_{ab})\,R] \mp M_{Fa} \quad \text{.........(28)}$$

and,

$$M_{ba} = S_b\,[\theta_b + C_{ba}\,\theta_a - (1 + C_{ba})\,R] \pm M_{Fb} \quad \text{.........(29)}$$

in which C_{ab} and C_{ba} are factors to carry over the moment applied at Ends a and b, respectively, to the other ends, b and a, which are assumed fixed; S_a and S_b are stiffness factors when a unit moment is applied at the designated end with the other end fixed; and M_{Fa} and M_{Fb} are fixed-end

NOTE.—The paper by John B. Wilbur, Assoc. M. Am. Soc. C. E., was published in December, 1935, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: March, 1936, by Messrs. C. A. Willson, Paul Andersen, and R. W. Stewart; and April, 1936, by Adolphus Mitchell, Jun. Am. Soc. C. E.

[17] Prof. of Structural Eng., Dept. of Civ. Eng., National Tsing Hua Univ., Peiping, China.

[17a] Received by the Secretary March 17, 1936.

[18] "The Modified Slope-Deflection Equations", by L. T. Evans, *Journal*, Am. Concrete Inst., October, 1931, p. 109.

[19] "Slope-Deflection Equations for the Analysis of Rigid Frames with Varying Moment of Inertia", by Fang-Yin Tsai, The Science Repts., National Tsing Hua Univ., Peiping, China. Series A, Vol. II, p. 75, July, 1933.

[20] "Analysis of Continuous Frames by Distributing Fixed-End Moments", by Hardy Cross, *Transactions*, Am. Soc. C. E., Vol. 96 (1932), p. 1.

moments at the designated ends, due to loading, with both ends fixed. For a member of which the moment of inertia varies symmetrically with respect to its center line:

$$C_{ab} = C_{ba} = C \qquad (30)$$

and,

$$S_a = S_b = S \qquad (31)$$

For a member of constant cross-section,

$$C = \frac{1}{2} \qquad (32)$$

and,

$$S = 4\,E\,K \qquad (33)$$

in which E and K are the same as the notation of the paper. With the values of C and S from Equations (32) and (33) substituted in Equations (28) and (29), the well-known ordinary slope-deflection equations for the case of constant moment of inertia are derived.

It may be noted that, according to the usual sign convention for moment adopted in the slope-deflection method, the carry-over factors, C_{ab} and C_{ba}, will always be positive, which is just opposite to that adopted in the method of moment distribution.

The determination of the values of the various constants in Equations (28) and (29) for members with moments of inertia varying in any manner and under any loading, the method of moment area, as shown by G. E. Large[21], Assoc. M. Am. Soc. C. E., will be found the most expedient. For certain special cases, there are many sets of tables and diagrams which give directly the values of those constants. Hence, with Equations (28) and (29) available, the application of this method to the case of variable moment of inertia will involve no difficulty or inconvenience.

A. Floris,[22] Esq. (by letter).[22a]—This ingenious method of analyzing statically indeterminate structures by means of the slope-deflection equation reduces, considerably, the number of the unknowns arising in such problems. It is elegant, direct, and does not involve approximations. Thus far, there are several methods available which render the analysis of structures with high redundancy comparatively easy. From the point of view of practical expediency simultaneous equations should be avoided, or their solution, at least, should be facilitated by special devices.

The avoidance of simultaneous equations by means of the moment distribution leads to tedious and indirect, if not difficult, processes. On the other hand, the method of iteration applied to the solution of simultaneous equations of a special kind, although convenient, is nevertheless time-consuming.

The author's method is a valuable contribution to the analysis of complicated structures. In principle, it is preferable to many widely used

[21] *Bulletin No. 66*, Ohio State Univ., Columbus, Ohio, November, 1932.

[22] Dipl.-Ing., Los Angeles, Calif.

[22a] Received by the Secretary March 20, 1936.

methods. However, whether it will replace them in practice is another matter. The question can be decided only by actual experience or comparative calculations.

A. W. FISCHER,[23] ESQ. (by letter).[23a]—The author has contributed a method for solving indeterminate structures by the slope-deflection method by what he terms "successive elimination", which seems to solve the simple examples he gives in a very short time. For solving secondary stresses in trusses by the slope-deflection method, however, the writer doubts if the author's method is any shorter than the use of a systematic table for solving the simultaneous equations.

Much has been written concerning the solution of rectangular frames by the slope-deflection method, but not much has been offered in regard to the solution of a symmetrical frame with inclined legs, supporting unsymmetrical loads. Consider, for example, the top story of the Kinzua Viaduct[24] with a horizontal load at the top (see Fig. 7). Assume the bases to be fixed and then compare the results with those that are obtained by the method of least work, which gives correct results, This bent is symmetrical and, using the author's method, sufficient equations can be written to solve it.

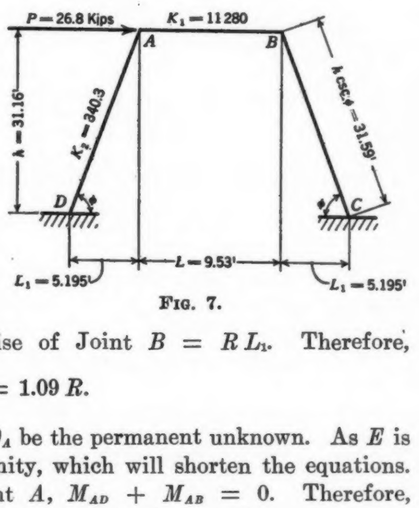

FIG. 7.

The change of length of members is considered zero; and, since the bent is symmetrical, $\theta_A = \theta_B$, $\theta_C = \theta_D = 0$. The horizontal movement at the top of Members AD and $BC = Rh$, and is to the right; the fall of Joint $A = R\,L_1$; and the rise of Joint $B = R\,L_1$. Therefore, the R-value of $A\,B = \dfrac{2R\,L_1}{L} = \dfrac{10.39}{9.53} = 1.09\,R$.

Of the two unknowns, θ_A and R, let θ_A be the permanent unknown. As E is constant, it can be assumed equal to unity, which will shorten the equations. For the condition, $\Sigma M = 0$, at Joint A, $M_{AD} + M_{AB} = 0$. Therefore, $2 \times 340.3\ (2\,\theta_A - 3\,R) + 2 \times 11\,280\ (3\,\theta_A + 3.27\,R) = 0$; from which,

$$R = -\,0.9625\ \theta_A \qquad \text{(34)}$$

A second true equation can be developed by considering the equilibrium of Member AB. Let s_1 and s_2 be the direct stresses of Members AD and BC; then:

$$s_1 \sin\phi + \frac{680.6\ (3\,\theta_A - 6\,R)\cos\phi}{h\csc\phi} - \frac{22\,560\ (6\,\theta_A + 6.54\,R)}{L} = 0 \qquad \text{(35)}$$

[23] Care, Pennsylvania Sugar Co., Philadelphia, Pa.

[23a] Received by the Secretary April 7, 1936.

[24] "The Kinzua Viaduct of the Erie Railroad Company", by the late Carl Robert Grimm, M. Am. Soc. C. E., *Transactions*, Am. Soc. C. E., Vol. XLVI (1901), p. 21.

$$s_2 \sin \phi - \frac{680.6\ (3\,\theta_A - 6\,R)\ \cos \phi}{h\ \csc \phi} + \frac{22\,560\ (6\,\theta_A + 6.54\,R)}{L} = 0 \quad \ldots (36)$$

and,

$$s_1 \cos \phi - \frac{680.6\ (3\,\theta_A - 6\,R)\ \sin \phi}{h\ \csc \phi} - 26.8 - s_2 \cos \phi$$
$$- \frac{680.6\ (3\,\theta_A - 6\,R)\ \sin \phi}{h\ \csc \phi} = 0 \quad \ldots\ldots\ldots\ldots (37)$$

From Equation (35),

$$s_1 = \frac{22\,560\ (6\,\theta_A + 6.54\,R)}{L\ \sin \phi} - \frac{680.6\ (3\,\theta_A - 6\,R)\ \cos \phi}{h} \quad \ldots\ldots (38)$$

and, from Equation (36),

$$s_2 = \frac{680.6\ (3\,\theta_A - 6\,R)\ \cos \phi}{h} - \frac{22\,560\ (6\,\theta_A + 6.54\,R)}{L\ \sin \phi} \quad \ldots (39)$$

Substituting the values from Equations (38) and (39) in Equation (37) and reducing:

$$\frac{66.30\ (6\theta_A + 6.54\,R)\ \cos \phi}{L\ \sin \phi} - \frac{2\ (3\,\theta_A - 6\,R)}{h} - \frac{26.8}{680.6} = 0 \quad \ldots (40)$$

Equation (40) also expresses the simple relation:

$$2\,M_{AD} + 2\,M_{DA} + \frac{4\ L_1}{L}\ M_{AD} + 26.8\ h = 0 \quad \ldots\ldots\ldots (40a)$$

Substituting the value of R from Equation (34) and $\sin \phi$ and $\cos \phi$ from Fig. 7, and reducing: $\theta_A = -0.04343$; and, then, from Equation (34), $R = +0.04180$.

From the foregoing values of θ_A and R:

$$M_{AD} = 2 \times 340.3\ (-0.08686 - 0.12540) = -144.5 \text{ ft-kips}$$

$$M_{AB} = 2 \times 11\,280\ (-0.08686 - 0.04343 + 0.13670) = 144.6 \text{ ft-kips}$$

and,

$$M_{DA} = 2 \times 340.3\ (-0.04343 - 0.12540) = -114.9 \text{ ft-kips}$$

For the upper story, with fixed bases, Mr. Grimm[24], found that,

$$M_{AB} = 4.765 \times 30.4 = +144.9 \text{ ft-kips}$$

$$M_{AD} = -144.9 \text{ ft-kips}$$

and,

$$M_{DA} = 9.69 \times 30.4 - 31.16 \times 13.4 = -114.7 \text{ ft-kips}$$

On comparing the foregoing results it can be seen that the two different methods practically check. (All calculations in this discussion were made with a 20-in. slide-rule.)

The general equations for solving a symmetrical bent with inclined legs, fixed bases, and various loads derived by "relative deflections"[25] and modified to solve the foregoing example are:

$$M_{AD} = -\frac{P\,L\,n\,h\,(3\,L + 4\,L_1)}{2\left\{L^2\,(1 + 6\,n) + 12\,L\,L_1\,n + 8\,n\,L_1^2\right\}} \quad\ldots\ldots\ldots\ldots(41)$$

and,

$$M_{DA} = -\frac{P\,L\,h\left\{L\,(1 + 3\,n) + 2\,L_1\,n\right\}}{2\left\{L^2\,(1 + 6\,n) + 12\,L\,L_1\,n + 8\,n\,L_1^2\right\}} \quad\ldots\ldots\ldots\ldots(42)$$

in which $n = \frac{K_1}{K_2}$. Substituting the values given in Fig. 7 in Equations (41) and (42), and reducing: $M_{AD} = -144.7$ ft-kips; and, $M_{DA} = -115.1$ ft-kips. These values, again, practically agree with those determined by Mr. Grimm[24].

If $L_1 = 0$, Equations (41) and (42) can be used to solve a symmetrical rectangular bent fixed at the bases and a horizontal load, P, at the top.

The author's method can be used for solving many problems in indeterminate structures, but to the writer it is just one tool, of many, which solves such simple examples as those given by the author in a very short time. A problem such as the analysis of a Vierendeel truss, with inclined top chord, however, is not so simple, and if the author would analyze a three-panel Vierendeel truss with an inclined top chord, the analysis might supply additional data for solving indeterminate structures with inclined members.

L. E. Grinter,[26] Assoc. M. Am. Soc. C. E. (by letter).[26a]—There is no question but that the author has made an excellent case for his procedure of automatically eliminating the unknowns in the slope-deflection equations. In fact, Professor Wilbur has hit upon a major reason why the slope-deflection method has lost some of its original popularity. The lack of a standard procedure for solving the simultaneous equations involved had led to considerable dissatisfaction with the method itself. The truth is that a confused sign convention and the difficulties involved in the solution of the simultaneous equations are the major objections that have been offered to its use. A physical picture of the relation of moment, sign, and curvature will eliminate the confusion in regard to signs, and the automatic procedure of eliminating unknowns suggested by Professor Wilbur will aid in simplifying the solution of the simultaneous equations.

Physical Significance of Signs.—The question of signs naturally is involved in the illustrative example presented in the paper. This example ends with the statement, "and the moment at the column base is $M = 2\,E\left(\frac{40}{2\,E} - \frac{160}{2\,E}\right) = -120$ ft-lb, the minus sign denoting that the couple, M, acts contra-clock-

[25] "Stresses in Statically Indeterminate Structures", by Prof. H. Yu, National Wuhan Univ., Wuchang, Hupeh, China, Second Edition, 1935, pp. 471–474.

[26] Prof. of Structural Eng., Agri. and Mech. Coll. of Texas, College Station, Tex.

[26a] Received by the Secretary April 10, 1936.

wise on the column." This statement without further explanation will not be understood by many readers. The signs of the end moments of an unloaded member will be made clear by Fig. 8. Such end moments are calculated by the standard slope-deflection equation, $M_{AB} = 2\,E\,K\,(2\,\theta_A + \theta_B - 3\,R)$. Evidently, therefore, the sign of the moment will be dependent upon the signs and relative magnitudes of θ_A, θ_B, and R.

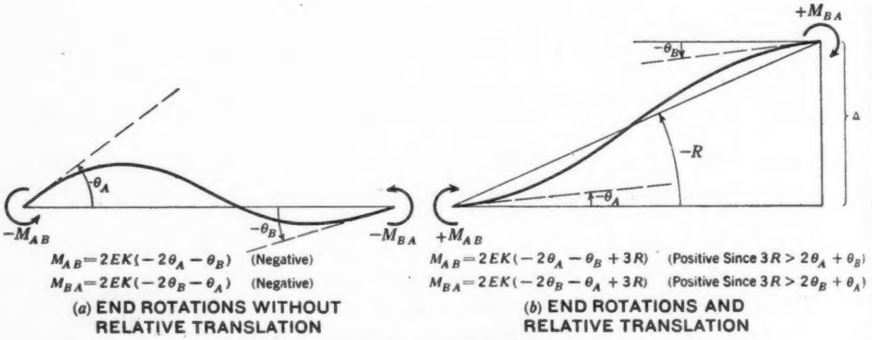

FIG. 8.—SIGNS OF END MOMENTS.

For the ordinary case of reversed curvature shown in Fig. 8(*a*), in which the value of R is zero, the sign of M is the same as the sign of the end slopes, θ_A and θ_B. The end rotations are counter-clockwise (negative) for the case illustrated, so that both the end moments, M_{AB} and M_{BA}, must be negative. Accordingly, the end moments represented by M_{AB} and M_{BA} must be the negative resisting moments external to the beam itself. Similarly, in Fig. 8(*b*), the geometry of the configuration determines the signs of the end moments. For the configuration shown, the value of $3\,R$ must be larger than $2\,\theta_A + \theta_B$ or $2\,\theta_B + \theta_A$. This is merely a geometric fact that any one can prove by attempting to re-sketch the diagram, assuming the reversed inequality. (For instance, when $3\,R = 2\,\theta_A + \theta_B = 2\,\theta_B + \theta_A$, the elastic line would be straight and the end moments would be zero.) Hence, for the case illustrated in Fig. 8(*b*), the end moments are of the same sign as the sign of R in the slope-deflection equations; that is, positive. Again, these end moments must be the resisting moments that act externally to the beam itself in order to agree with these signs.

Advantages of a Routine Procedure.—The usefulness of Professor Wilbur's procedure for the successive elimination of unknowns will depend very largely upon the number of simultaneous equations involved in the standard solution; that is, upon the number of unknowns. The continuous beam illustrated in Fig. 1(*a*) of the paper is indeterminate only to the third degree. The three simultaneous equations in any standard analysis can be solved so simply and conveniently that the successive elimination of unknowns would offer little advantage. However, the wind frame of Fig. 1(*b*) is a highly indeterminate structure the analysis of which would require the solution of twenty simultaneous equations, a tedious and wholly objectionable task. Certainly, there

could be no justification of the use of the slope-deflection method for the analysis of this frame, unless some automatic procedure, such as that suggested by the author, was to be used.

Undoubtedly, many engineers have been applying the procedure suggested by the author, but in a rather haphazard fashion. The writer has found his students quite confused by the fact that different members of the class would seem to obtain the solution for a problem such as Fig. 1(*a*) by solving different numbers of simultaneous equations. In effect, they were applying the suggestions of the author haphazardly. The writer, therefore, feels that Professor Wilbur has clarified the solution of the slope-deflection equations, and that his contribution is of considerable importance.

AMERICAN SOCIETY OF CIVIL ENGINEERS

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DISCUSSIONS

CONSERVATION OF WATER

PROGRESS REPORT OF THE COMMITTEE OF THE IRRIGATION DIVISION

Discussion

BY MESSRS. W. P. ROWE, AND A. A. YOUNG

W. P. ROWE,[11] ASSOC. M. AM. SOC. C. E. (by letter).[11a]—This report is divided into fourteen subjects most of which the writer will discuss in the order presented. The Bureau of Agricultural Engineering of the U. S. Department of Agriculture, under the direction of Harry F. Blaney, M. Am. Soc. C. E., has developed a method for determining precipitation absorption in the valley areas. This work should be extended to include the mountain areas. The writer's experiences lead him to believe that there may be appreciable precipitation penetration to considerable depths in certain mountain areas, particularly in faulted regions. The San Jacinto and Val Verde Tunnels, of the Metropolitan Water District of Southern California, the Gibraltar Tunnel, of the Santa Barbara Water Department, and numerous other tunnels in Southern California, offer an ideal opportunity to study this phase of the problem. The problem of precipitation absorption on mountain water-sheds is also one of great importance in the interpretation of snowfall data.

Evaporation.—The principal need for additional evaporation data is in regions that are susceptible to future reclamation for agricultural purposes. So many valuable evaporation data have been assembled and published by the Society and other agencies that a great mass of additional data would only increase the confusion. The paper, by Carl Rohwer[12], Assoc. M. Am. Soc. C. E., is a classic in evaporation investigation.

NOTE.—The Progress Report of the Committee of the Irrigation Division on the Conservation of Water, was presented at the meeting of the Irrigation Division at Los Angeles, Calif., July 4, 1935, and published in December, 1935, *Proceedings*. Discussion on the report has appeared in *Proceedings*, as follows: April, 1936, by Messrs. Donald M. Baker, and Dean C. Muckel.

[11] Cons. Engr., San Bernardino, Calif.

[11a] Received by the Secretary March 7, 1936.

[12] "Evaporation from Free Water Surfaces", by Carl Rohwer, *Technical Bulletin No. 271*, U. S. Dept. of Agriculture, in connection with Colorado Agricultural Experiment Station.

There is need for the adoption of a standard pan and setting to insure the continuance of data at existing stations. The correlation of pan data to reservoir or lake surfaces is as consistent as any engineer needs in planning his work, but there is need for a correlation between evaporation pan data and transpiration losses from vegetated areas so that these losses may be estimated more closely. The writer knows of no reservoir sites in Southern California that were not utilized solely because of lack of knowledge of the rate of evaporation. As an evaporation record is much more consistent from year to year than a rainfall record, the estimation of the loss within 6 in. per yr is close enough for most engineering studies.

Dr. N. W. Cummings, of San Bernardino, Calif., has done considerable research work on an evaporation formula which embodies the effects of solar energy[13]. He has apparently evolved a formula which can be used as a precise measurement of evaporation. The chief application of such a formula would be in the determination of hidden losses from reservoirs which might not be detected under the present methods of determining evaporation losses by correlation of pan and reservoir data. A precise determination of evaporation would also be of great value in making studies of the effects of storage losses on the increase in salinity of waters which are already of relatively high saline content.

Economic Use of Irrigation Water.—Extensive studies of this subject have been conducted by Federal and State agencies for many years. The practical application of this knowledge has led to appreciable savings in power and water costs by irrigators, but as a means of conserving water in most Southern California areas it is of minor importance. Most of the soils underlying the present irrigated area, the principal exception being those near the ocean, are sufficiently open to transmit readily any irrigation water in excess of surface evaporation and plant requirements to the water-table where it again becomes available for re-use. There is a tendency for this return irrigation water to increase in salinity by re-use, but thus far, any deterioration in the quality of this water supply in the South Coastal Basin of Southern California from this phenomenon, has been overcome by mixing this supply with less saline waters. The importation of a new supply of water, of a quality very close to the most saline now being used, will certainly upset the "physiographic balance" now existing.

The application of just enough irrigation water of relatively high salinity to maintain soil moisture above the field capacity only within the root zone will result in the accumulation of salts which eventually will be harmful to the growing plant. The quantity of excess water to be applied to prevent such accumulation is a problem for the soil chemist, but is closely allied with irrigation engineering and must be considered by the Engineering Profession. One of the features of the State Water Plan for the Central Valley Water Project[14] provides for the application, during wet years, of quantities of

[13] "Relation Between Evaporation and Humidity", by N. W. Cummings, *Bulletin No. 68*, National Research Council (1929); also, "Evaporation from Lakes", by N. W. Cummings and B. Richardson, *Physical Review*, Vol. 30 (1927).

[14] Reports on State Water Plan, *Bulletin No. 29*, "San Joaquin River Basin," 1931, Publications of the Diversion of Water Resources, Dept. of Public Works, State of California, pp. 333 *et seq.*

irrigation water far in excess of estimated consumptive requirements for the purpose of replenishing ground-water supplies depleted by pumping during dry years. This a reversal of the prevailing trend of agricultural practice and can be largely avoided if the main and branch canals are left unlined as at present.

Studies and investigations by the U. S. Bureau of Agricultural Engineering indicate that the climate of Southern California, with erratic periods of low humidity and high evaporation, requires the application of irrigation water on short notice if the best crop returns are to be obtained. This irrigation practice is increasing and requires a large stand-by of water either in surface reservoirs or in underground storage. With the general adoption of such practice an increase in the consumptive use by irrigated vegetation may be expected.

Soil Erosion.—Since erosion is at the mercy of the whims of the weather, and since the Soil Conservation Service will control erosion, it seems that the hope of Mark Twain is finally to be realized. One fertile field for this Service is the prevention of erosion from the so-called "fire-breaks" constructed by the U. S. Forest Service. The prevention of erosion from these denuded strips, each of which supports an erosion channel, is dependent upon the re-growth of the original species of brush existing before their construction. The elimination of these æsthetic monstrosities, however, will remove the picturesque "razzle-dazzle", jig-saw puzzle background of the Southern California valleys and eliminate the greatest outdoor advertising medium ever conceived by a Federal agency. This agency will no doubt oppose any such obliteration of its handiwork.

Burning of Native Vegetation.—There is little need for the discussion of this subject. Under the previously prevailing practice of the Forest Service, it was amply demonstrated that this native vegetation would burn. The writer has been critical of the Forest Service during recent years in its handling of such fires and the incidental propaganda and post-mortems after each blaze. However, the recent adoption of controlled burning (backfiring)[15] as a means of preventing the spread of chaparral fires has eliminated his chief criticism. The Service has demonstrated that burning can be controlled under the most adverse conditions of high winds, low humidity, and minimum soil moisture, and it only remains to adopt this policy under more favorable conditions. The writer suggests that this method of controlled burning be practiced in the spring, when soil moisture is at a maximum and is enough to cause the burned brush to coppice sufficiently to present a water-shed cover before the winter floods occur.

When Joyce Kilmer wrote "Only God can make a tree", he reckoned without the arboriculturists of the Forest Service. They did not have to make a tree. They took the native chaparral of California, called it a forest, and extended their dominion over a larger realm. At one time, the Forest Service enjoyed the confidence of every camper and Nature lover in Southern

[15] "Backfiring on the Malibu Fire", *Conservation Activities*, Conservation Assoc. of Los Angeles County, November, 1935, p. 4.

California. The policing of this chaparral area and the restrictions put on visitors to the mountains have alienated this former high regard.

The writer hopes that the Forest Service will relinquish to some other Federal or State agency its dominion over weed and brush and return to the field for which it was originally created, namely, the opening of forests to the public use. The Forest Service has demonstrated that it can go into an area with heavy chaparral cover, select a portion of this area for experimental purposes, and burn off the brush under control to the exact boundaries desired. The extension of this technique to other areas would be a long step toward the prevention of mass burning with its resultant maximum débris movement during the first rain, but the adoption of such a policy would have to be made by a different agency from the one which has been so vehemently opposed to the suggestion.

The Forest Service is spending hundreds of thousands of dollars in gathering climatological and hydrological data in an experimental area in Southern California. The Society would be doing a public service if it reviewed the enormous mass of data being gathered in this study and rendered an impartial report of its findings. The writer believes that many data will be obtained which will have no particular significance to the forester, but which will be valuable to the engineer studying water supply problems. These data should be made available to the Engineering Profession.

Transportation of Débris.—This is a study that is "between two fires." There are the soil erosion faddists on one side, showing how much débris flows off the mountains with every fog, and the water spreaders who with an abhorrence of any water "wasting into the ocean", try to put underground as soon as possible, the transporting medium for the débris. When bigger and better fires occur, when bigger and better débris flows are measured, and when bigger and better water-spreading works are constructed, one may look for the mountains to be leveled, the valleys filled, and the geologic norm attained.

Water-Spreading and Flood Channels.—The spending of millions of dollars in water-spreading works on Southern California streams during the present dry period has built up a false feeling of security in their adequacy as flood-control measures. The recent depression had a great retarding effect on the movement of real estate in Southern California, but with a return to normalcy a movement of population can be expected to cheap home sites which are only available in hazardous locations. The proper functioning of a planning commission, whether State or County, would prevent the creation of a decided menace along these lines. Adequate flood channels with stable bank protection are absolutely necessary in Southern California. Many of the water-spreading works have been constructed without regard to silt deposition, and many more have been constructed which turn the silt-laden tail-water back into a formerly porous stream channel which is soon rendered impervious. The periodic sluicing of the absorption areas is essential if their porosity and absorptive powers are to be maintained. This, in itself, calls for the maintenance of these flood channels.

Check Dams.—The behavior of check dams during the Montrose-La Crescenta flood has demonstrated that, in Southern California, dams constructed

as these were, are not worthy of the name. In this instance, they were a positive menace, first by creating a false sense of security and then causing the diversion of flood and débris flows to lands that would otherwise have been unharmed. A properly constructed check dam used in stabilizing the existing grade of a gully or a stream is of great value, but when it is used to raise the grade of a stream channel it can become a menace to life and property. The precipitous grades of most of the small streams in Southern California on which check dams have been used, have caused their condemnation. It is probable that they can be used effectively in regions that are not subject to such torrential floods as Southern California experiences.

The type of débris basin having a protected inlet to prevent a retrogression in grade of the entering stream, as now being built by the Los Angeles County Flood Control District, offers all the protection formerly claimed for check dams, and at less expense. The Committee is to be congratulated for the resolution it presented to the Conference of 1935 which was unanimously adopted and is included in the report under discussion.

Investigations in Water Conservation and Control.—The Committee correctly states that since most of the water conservation works for spreading water have been constructed during the recent dry period and have never been tested by a flood, some continuance of studies on the cost of operation should be made. Such studies can only reveal that most of these works are not economically sound. The engineers in charge of this class of construction should not be criticized for the apparent cost of these works. The policy of the Federal Government in putting unemployed labor to work on constructive projects rather than on weed cutting and leaf raking activities, made available vast sums of money for this type of construction. Plans had to be drawn hastily and put into operation with no assurance that the flood of labor and money for materials would be forthcoming for any length of time.

This policy of the Federal Government resulted in a great acceleration in the construction of spreading works and projects, which under ordinary conditions would have been constructed from small annual appropriations over a great many years of trial. As each water-shed and stream has its own peculiar botanic, climatic, topographic, and geologic characteristics, these spreading works are of many different types. Because no flood flows have occurred that would test the effectiveness of the various types of structures, and because there is no assurance that this method of employing idle labor will be discontinued, it would seem that a further study of the operations of existing structures under the prevailing method will result in the design of more efficient works for other localities in the future.

The U. S. Bureau of Agricultural Engineering, under the direction of Mr. A. T. Mitchelson, is engaged in correlating data on the efficiency of the various water-spreading works. This investigation should be continued until a capital flood offers the opportunity for testing the present works thoroughly.

The Committee is to be complimented on its handling of problems so vital to Western water supply. The printing of all the papers mentioned in the report in a single volume would make a valuable addition to any library.

A. A. YOUNG,[16] Assoc. M. Am. Soc. C. E. (by letter).[16a]—As it refers to evaporation research, this report suggests further investigations combining the points of view of the engineer and the physicist in approaching the evaporation problem. This suggestion is to be commended as there is increasing evidence that both views must be considered. Neither one, at present, is entirely satisfactory, although much that is of value has been accomplished to accord with both. It seems likely that earnest co-operation by engineers and physicists, combining the methods that each have developed, will produce results of greater value than any that have yet been obtained.

Much work has been done by physicists in determining the close relation existing between insolation, or "exposure to the sun's rays", and evaporation. The effect of insolation may be measured by means of a recording pyrheliometer. Unfortunately, few stations exist at which such records are available, and when obtained they are seldom applicable to other localities.

For years engineers have studied evaporation through measurement of losses from pans of various sizes and types, and have developed coefficients for reducing such losses to lake or reservoir values. The Division of Irrigation, Bureau of Agricultural Engineering, U. S. Department of Agriculture, has done pioneer work along this line since the turn of the century first showed the necessity of conserving the water supplies in the West. As early as 1903, evaporation studies were made in Southern California under the direction of the late Samuel Fortier, M. Am. Soc. C. E.[17]

Later, the Division undertook the important study at Denver, Colo., under the direct charge of the late R. B. Sleight, Assoc. M. Am. Soc. C. E., which developed the first satisfactory coefficients for reducing pan evaporation to reservoir equivalents.[18] Additional studies at the same station were made in 1919 by Harry F. Blaney, M. Am. Soc. C. E. The largest evaporation pan used at this time was of the type adopted as a standard by the Bureau of Agricultural Engineering, 12 ft in diameter by 3 ft deep.

To check coefficients developed at Denver, additional studies were made at Fort Collins, Colo., by Carl Rohwer, Assoc. M. Am. Soc. C. E., under the direction of the Division, and a report was published in 1931.[19] In this investigation a copper-lined reservoir 85 ft in diameter was used instead of the 12-ft pan which was the basis of the Denver measurements. Sleight[18] had concluded that evaporation from a 12-ft pan was nearly equal to that from a larger body of water, and this conclusion was concurred in by Rohwer.[19]

The Committee recommends the selection of some place in California for further study of evaporation because it "offers favorable climatic conditions with its freedom from winter freezing and the large rates of loss during summer months." The writer's opinion agrees with this statement. Cali-

[16] Asst. Irrig. Engr., Div. of Irrig., Bureau of Agricultural Eng., U. S. Dept. of Agriculture, Pomona, Calif.

[16a] Received by the Secretary March 20, 1936.

[17] "Evaporation Losses in Irrigation and Water Requirements of Crops", by Samuel Fortier, *Bulletin 177*, O. E. S., U. S. Dept. of Agriculture.

[18] "Evaporation from Surfaces of Water and River-Bed Materials", by R. B. Sleight, *Journal of Agricultural Research*, Vol. 10, No. 5, 1917.

[19] "Evaporation from Free Water Surfaces", by Carl Rohwer, *Technical Bulletin 271*, U. S. Dept. of Agriculture, 1931.

fornia is representative of a large geographical area in the Pacific Southwest where seasonal evaporation exceeds that for the inter-mountain country. Not only is the daily evaporation high, but the evaporating season is continuous throughout the year.

In Southern California the Division of Irrigation, in co-operation with other agencies, has been keeping evaporation records since 1928. The first of these records was in connection with consumptive use of water studies by native vegetation[20], but in 1932 a co-operative evaporation study was undertaken at Baldwin Park, Calif., to determine the relationships existing between different types of evaporation pans in common use. Records of evaporation at this station have been continuous from July, 1932, to the present time. A progress report was published in 1933.[21]

Further investigation was begun in January, 1935, by the Division of Irrigation and other agencies at Fullerton on the Coastal Plain of Southern California. In this connection opportunity exists for intensive investigations under coastal climatic conditions. Only ordinary evaporation equipment is now used, but it is hoped that eventually complete equipment for insolation studies, including insulated pans, may be installed. The scarcity and value of water in the Pacific Southwest and the interest in evaporation shown by physicists and engineers make this location particularly suitable for further investigations.

TABLE 1.—DESCRIPTION OF EVAPORATION PANS

Pan No.	Diameter, in feet	Depth of pan, in feet	Depth of water, in feet	Depth set in the ground, in feet	Type of evaporation pan
(*a*) AT BALDWIN PARK, CALIFORNIA					
1.....	4	0.83	0.62	Above	U. S. Weather Bureau
2.....	6	3.0	2.75	2.75	U. S. Bureau of Agricultural Engineering
3.....	2	3.0	2.75	2.75	Los Angeles County Flood Control District
4.....	2	3.0	2.75	2.75	U. S. Bureau of Agricultural Engineering
(*b*) AT FULLERTON, CALIFORNIA					
1.....	12	3.0	2.75	2.75	U. S. Bureau of Agricultural Engineering
2.....	6	3.0	2.75	2.75	U. S. Bureau of Agricultural Engineering
3.....	3*	1.5	1.17	1.17	Colorado
4.....	2	3.0	2.75	2.75	U. S. Bureau of Agricultural Engineering
5.....	1	3.0	2.75	2.75	U. S. Bureau of Agricultural Engineering
6.....	4	0.83	0.62	Above	U. S. Weather Bureau
7.....	2	0.83	0.62	Above	....
8.....	2.5*	0.71	0.62	Above	Insulated

* Square pan

The Baldwin Park Station has four types of evaporation pans as described in Table 1(*a*). In addition, the meteorological equipment includes maximum and minimum thermometers, rain-gage, anemometer, atmometer, hygrothermograph, and barograph. Floating maximum and minimum thermometers have

[20] "Rainfall Penetration and Consumptive Use of Water in the Santa Ana River Valley and Coastal Plain", *Bulletin No. 33*, California Dept. of Public Works, 1930.

[21] "Water Losses under Natural Conditions from Wet Areas in Southern California", *Bulletin No. 44*, California Dept. of Public Works, 1933.

been maintained in each evaporation pan although records of water temperatures are not complete. Monthly evaporation records for the period, July, 1932, to December, 1935, inclusive, are given in Table 2.

The Fullerton Station is larger and better equipped than the station at Baldwin Park. The various pans are listed in Table 1(*b*). Additional pans

TABLE 2.—EVAPORATION RECORDS, BALDWIN PARK STATION, CALIFORNIA (ELEVATION, APPROXIMATELY 400 FEET)*

Month and year	Evaporation from Pans, in Inches				Ratios (Percentages)		
	U.S. Weather Bureau pan, 4 ft in diameter, 10 in. deep; No. 1	U. S. Bureau of Agricultural Engineering pan, 6 ft in diameter, 3 ft deep;† No. 2	Los Angeles County Flood Control District pan, 2 ft in diameter, 3 ft deep;‡ No. 3	U. S. Bureau of Agricultural Engineering pan, 2 ft in diameter, 3 ft deep; No. 4	Pan No. 2 to Pan No. 1	Pan No. 3 to Pan No. 1	Pan No. 4 to Pan No. 1
1932:							
July	8.30	7.47	9.31	....	90.0	112.2	....
August	8.02	7.26	9.40	....	90.5	117.2	....
September	5.64	4.81	6.57	....	85.3	116.5	....
October	5.00	4.43	5.63	....	88.6	112.6	....
November	4.23	4.06	4.81	....	96.0	113.7	....
December	2.07	2.22	2.37	....	107.2	114.5	....
1933:							
January	2.47	1.89	2.21	....	76.5	89.5	....
February	3.49	2.38	3.15	....	68.2	90.2	....
March	4.79	3.61	4.88	....	75.4	101.9	....
April	5.28	4.16	5.83	....	78.8	110.4	....
May	6.89	6.43	7.75	....	93.3	112.5	....
June	8.15	6.89	9.09	....	84.5	111.5	....
July	9.49	7.75	10.04	4.73§	81.7	105.8	....
August	8.53	6.87	9.41	8.49	80.5	110.3	99.5
September	5.68	4.95	6.14	5.74	87.1	108.1	101.1
October	4.84	4.04	5.02	4.63	83.5	103.7	95.7
November	4.16	3.13	4.13	3.93	75.2	99.3	94.5
December	‖	‖	‖	‖	....	....	....
1934:							
January	2.74	1.79	1.95	1.99	65.3	71.2	72.6
February	2.38	1.57	1.65	1.80	66.0	69.3	75.6
March	4.58	3.59	4.35	4.12	78.4	95.0	90.0
April	5.99	5.05	6.38	5.97	84.3	106.5	99.7
May	8.39	7.23	9.06	8.52	86.2	108.0	101.5
June	6.41	5.46	6.84	6.54	85.2	106.7	102.0
July	9.44	7.78	9.79	9.32	82.4	103.7	98.7
August	8.32	7.11	8.89	8.30	85.5	106.8	99.8
September	7.32	6.04	7.71	7.21	82.5	105.3	98.5
October	4.50	3.52	4.40	4.25	78.2	97.8	94.4
November	2.53	2.17	2.77	2.69	85.8	109.5	106.3
December	2.11	1.65	2.10	2.17	78.2	99.5	102.8
1935:							
January	2.18	1.52	1.95	1.90	69.7	89.4	87.2
February	2.94	2.23	2.86	2.64	75.8	97.3	89.8
March	3.56	2.71	3.38	3.26	76.1	94.9	91.6
April	4.72	3.78	4.54	4.33	80.0	96.2	91.7
May	5.94	4.91	6.38	5.70	82.6	107.4	96.0
June	7.50	6.24	8.54	7.14	83.2	113.9	95.2
July	9.38	7.76	10.17	9.13	82.7	108.4	97.3
August	8.82	7.22	9.64	8.59	81.8	109.3	97.4
September	6.74	5.54	7.46	6.49	82.2	110.7	96.3
October	5.37	4.44	5.96	5.13	82.7	111.0	95.5
November	3.19	2.50	3.42	2.99	78.4	107.2	93.7
December	2.38	1.84	2.46	2.19	77.3	103.4	92.0

* This table is published in part in the Rept. on Progress Conference on Water Conservation, Los Angeles, Calif., March 13–14, 1935, by the Committee on the Conservation of Water, Irrigation Division, Am. Soc. C. E., December, 1935.

† Type used by San Gabriel Protective Assoc. in valley areas.

‡ Type used by Los Angeles County Flood Control Dist. in mountain areas

§ For period, July 15 to July 31, inclusive.

‖ Pans overflowed during storm; record incomplete.

will be added as opportunity permits. The meteorological equipment includes maximum and minimum thermometers, standard rain-gage, recording rain-gage, anemometer, thermograph in the thermometer shelter, thermograph for sun temperatures at water-surface levels, and distance thermograph for recording continuous water temperatures in evaporation pan. Maximum and minimum thermometers are also floated on each pan.

An analysis of the results at Baldwin Park shows a wide variation in ratios of evaporation from different pans within short periods of time, sometimes amounting to as much as 100% in 24 hr. As the length of the evaporation period increases, the average ratios are gradually smoothed out although slight changes are still occurring after $3\frac{1}{2}$ yr of measurements. Monthly ratios of evaporation from the 6-ft pan to the standard pan of the U. S. Weather Bureau, have been plotted in Fig. 1, together with corresponding mean monthly values of humidity, wind velocity, and air temperature.

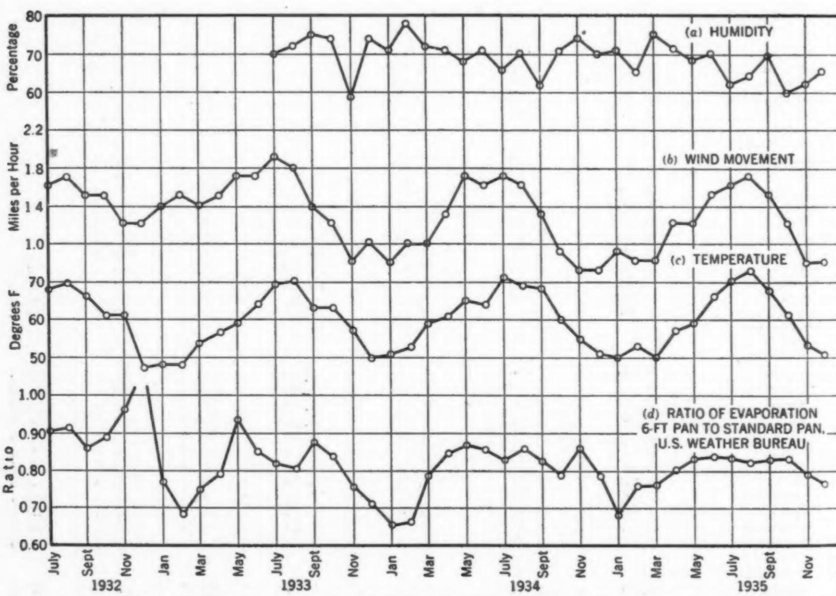

FIG. 1.—MEAN MONTHLY VALUES, BALDWIN PARK EVAPORATION STATION, CALIFORNIA.

The ratio curve brings out the important fact, previously given little attention, that ratios for the winter months are consistently less than those for the summer. This difference has been noted in a number of places in Southern California. The mean monthly ratios as plotted are influenced by air temperature and wind velocity, but not by humidity. It is a characteristic of the coastal climate at Baldwin Park that humidity normally goes through a daily cycle of from nearly 100% at night to 30 to 40% during the day. This is almost a daily occurrence, regardless of season; hence there is no marked seasonal change in the mean monthly value.

The lower ratios in the winter are consistent for each year of record. Inconsistencies in daily and monthly evaporation have been noted by others, but because previous investigations have been made in Colorado where winter measurements are impossible, low winter ratios, as obtainable in California, have not been observed. The coefficient or ratio of 0.70 developed in Colorado for the reduction of evaporation from a U. S. Weather Bureau pan to lake or reservoir losses is applicable to climates similar to that in which it was developed, but is about 10% less than the coefficient indicated by preliminary investigation in California. Further study is necessary before a definite value can be recommended for use in the Pacific Southwest.

The work done by Sleight[18] and Rohwer[19] was pioneering, but it provided reduction coefficients which have been accepted by the Engineering Profession. If the investigation in California results in slightly different values it will be because of difference in length of evaporating season and other climatic factors. It can reflect in no way upon previous studies.

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DISCUSSIONS

MODERN CONCEPTIONS OF THE MECHANICS OF FLUID TURBULENCE

Discussion

BY MESSRS. S. FRANZ YASINES, BENJAMIN MILLER, AND RALPH W. POWELL

S. FRANZ YASINES,[25] ASSOC. M. AM. SOC. C. E. (by letter).[25a]—An excellent résumé of the investigations that have been made in this field of fluid turbulence since the middle of the Nineteenth Century, is contained in this paper.

Because the current mathematical, as well as the physical, knowledge of many fluid-flow problems is limited, present-day investigators are forced to idealize the conditions in such problems, in order to save time and labor in actual research in the laboratories; and in this manner they find at least a partial solution of a problem in an empirical form. When an attempt is made to imitate Nature in the laboratory, in performing tests on models, the investigator frequently resorts to such guides as Froude's or Reynolds' numbers. Often he finds that they fail to perform their functions. Is this because their true meanings (and hence their limitations in application to actual problems) are not understood?

Some hydraulicians maintain that Reynolds' number is the ratio of inertial force to that of frictional force in a given flow. Such a definition is confusing when flow in a straight pipe of uniform cross-section is under consideration. The Newtonian principle of momentum may be applied to a flow confined in such a conduit, but at a constant rate of flow the rate change of momentum is zero; hence, the inertial force in this particular flow must also be zero. Therefore, if Reynolds' number is to be defined as the ratio of inertial force to frictional force, it must be zero for any constant rate of flow in a conduit of uniform cross-section.

The brief derivation of Reynolds' number, presented by the author, leads to the conclusion that the only forces governing the flow of fluids in pipes are frictional and a certain fictitious force which he terms "the rate of

NOTE.—The paper by Hunter Rouse, Assoc. M. Am. Soc. C. E., was published in January, 1936, *Proceedings*. Discussion on the paper has appeared in *Proceedings*, as follows: April, 1936, by Chesley J. Posey, Jun. Am. Soc. C. E.

[25] Instructor in Civ. Eng., New York Univ., New York, N. Y.

[25a] Received by the Secretary March 27, 1936.

passage of momentum." What is the latter force? Why not consider pressure force, since it is known definitely that the governing forces in the problem of motion under consideration are friction and pressure? The rigorous mathematical derivation of Reynolds' number is by no means easy when a physical conception of the problem is to be formulated. Considering, for example, the application of dimensional analysis (π-theorem) to the flow (at constant rate) in a conduit of uniform cross-section, it is possible to obtain the proper relation among the variables which indicates the presence of Reynolds' number, provided the independent variables are properly selected. On the other hand, how is one to know at the beginning of the analysis what variables are to be considered as being independent if no experimental data are available?

By using the Navier-Stokes equation of motion and the Newtonian principle of dynamic similarity, it can be proved that the ratio of inertial force to that of frictional force may be equal to zero or to unity (depending upon the assumption made in the physical analysis of the flow), and yet Reynolds' number does appear as a governing factor for similarity of flow.

The reliability of investigations of the boundary layer with a thickness of 0.001 ft, by means of specially designed Pitot tubes, is questionable. Instruments based on electrical rather than physical principles would be more reliable in the study of such complex problems.

The writer agrees with the author that a more specific definition of surface roughnesses should be established. This can be done since apparatus for this purpose is already available.

Since the contemporary hydraulician has inherited a large amount of information from his predecessors and has the opportunity of resorting to efficient physical tools for the study of various types of fluid flow, there is good reason to believe that he may solve many intricate problems; but, in order to do so, a closer co-ordination among hydraulicians and physicists is essential.

BENJAMIN MILLER,[26] ESQ. (by letter).[26a]—The study of turbulent flow in commercial pipe has been conducted along both analytical and empirical lines. The author has referred to work of both types, and gives the impression that whereas great strides have been made on the analytical side, the empirical studies have not been of much help, and that as yet the engineer cannot use the results of recent work for design. The writer feels that the picture is rather different, that the analytical methods described are far from satisfactory, but that fluid-flow rate can now be predicted with great accuracy by means of the empirical relationship given in the paper. That empirical relationship is due entirely to experiment, and owes nothing to analytical work. A rational analysis of turbulent flow should lead to the same relationship as that found empirically. The methods described in the paper do not, but improved methods of the future may.

[26] With Cities Service Co., New York, N. Y.

[26a] Received by the Secretary April 15, 1936.

Before giving reasons for these statements, the writer wishes to point out that the law of viscous flow was discovered by Hagen in 1839 and established beyond doubt by Poiseuille in 1842, but it was not until 1856 that Wiedemann derived the law analytically. The law of turbulent flow was first published in 1932, although experimental work was available previously on which it might have been based. Perhaps, by 1946, an analytical derivation of it may have been developed.

The law of turbulent flow to which the writer refers is the author's Equation (68). Mr. Rouse states that the appearance of the square root of f on both sides of the equation is a drawback to its use for general purposes; but this is a drawback only because of the habit of thinking in terms of f, which in turn, comes about through thinking of the pressure drop required to maintain a given flow rate. If, however, one thinks in terms of the flow rate which can be maintained by a given pressure drop, one may transform Equation (68) into,

$$\frac{m}{t} = \frac{\pi}{\sqrt{8}}\left(D^5 \rho \frac{dp}{dL}\right)^{0.5}\left(\log_{10} \frac{D^3 \rho \frac{dp}{dL}}{\mu^2} - 0.5\right) \ldots\ldots (71)$$

Introducing the following symbols:

For the flow rate,

$$Q = \frac{m}{\rho t} \ldots\ldots (72)$$

for the pressure gradient,

$$G = \frac{dp}{dL} \ldots\ldots (73)$$

and, for the von Kármán number[27],

$$K = \frac{D^{1.5} \rho^{0.5} G^{0.5}}{\mu} \ldots\ldots (74)$$

Equation (71) becomes,

$$Q = \frac{\pi}{\sqrt{2}}\left(D^5 \frac{G}{\rho}\right)^{0.5} (\log_{10} K - 0.25) \ldots\ldots (75)$$

Equation (75) is suitable for general use. It describes, accurately, the turbulent flow in commercial steel pipe. The flow through pipe lines will be less because of the pressure drops at joints, bends, fittings, etc.

The development of Equation (75) is detailed in the paper previously cited.[27] The author's Equation (8) may be used as a starting point, but it is written instead,

$$V = \phi_1 (D, \rho, \mu, \tau_0) \ldots\ldots (76)$$

Since Q is proportional to V,

$$Q = \phi_2 (D, \rho, \mu, \tau_0) \ldots\ldots (77)$$

[27] *Transactions*, Am. Inst. Chem. Engrs., Vol. 32 (1936), p. 1.

By dimensional analysis Equation (77) transforms to,

$$Q = \left(D^5 \frac{G}{\rho}\right)^{0.5} \phi_3 (K) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (78)$$

and Equation (78) is put into the explicit form of Equation (75) by the empirical method of plotting the results of the various investigators, such as Stanton and Pannell, Nikuradse, etc.

Equation (75) is represented well by the author's Equation (35) between Reynolds numbers of 10^5 to 10^7, but it is not restricted to this region. Actually, it coincides with experiment down to the beginning of the turbulent régime. As mentioned previously, it applies to pipe, and not necessarily to pipe lines. However, there are commercial pipe lines in which the flow is more than 95% of that calculated by Equation (75), and it is an unusual line in which the flow is less than 90% of that so calculated.

The author does not consider that the methods of analysis described in the paper are perfect. He does indicate that, although there is a fallacy in the fundamental assumptions, the discrepancy thus introduced is so slight as to be entirely negligible in practice. Rather than pick out the discrepancies, the writer suggests that the problem be re-examined in the light of what is known.

In turbulent flow, as in viscous flow, there must be maximum velocity at the center and zero velocity at the wall. Furthermore, there must be a viscous drag proportional to the rate of change of velocity.

These statements may be expressed as follows: $v = 0$ at $r = r_0$; $\frac{dv}{dr} = 0$ at $r = 0$; $\mu \frac{dv}{dr} = \tau_0$ at $r = r_0$; and,

$$\tau = \mu \frac{dv}{dr} + f(r) = \frac{\tau_0 r}{r_0} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (79)$$

The law of turbulent flow (Equation (75)), is:

$$2\pi \int_0^{r_0} r\, v\, dr = \frac{\pi}{\sqrt{2}} \left(D^5 \frac{G}{\rho}\right)^{0.5} (\log_{10} K - 0.25) \ldots\ldots (80)$$

Any analytical expression for v as a function of r must meet all these conditions. Assumptions which lead to infinite velocity gradient at the wall and infinite velocity at the center may be interesting, but they can not be correct.

Ralph W. Powell,[28] M. Am. Soc. C. E. (by letter).[28a]—The title of this paper is perhaps too broad, as it scarcely covers the entire subject of fluid turbulence. The question of turbulence as it affects problems in aeronautics[29] and the transportation of sediment in streams[30], for instance, is scarcely

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[28] Assoc. Prof. of Mechanics, Ohio State Univ. Columbus, Ohio.

[28a] Received by the Secretary April 22, 1936.

[29] See, for example, "Applied Hydro- and Aeromechanics", by Prandtl and Tietjens, Chapters IV, V, and VI, N. Y., McGraw-Hill Pub. Co., 1934.

[30] "Review of the Theory of Turbulent Flow and Its Relation to Sediment Transportation", by Morrough P. O'Brien, Assoc. M. Am. Soc C. E., *Transactions*, Am. Geophysical Union, April, 1933, pp. 487–491.

touched upon. As a résumé of present knowledge on the flow in pipes, however, the paper is excellent. The only other adequate treatment of the subject in English, as far as the writer knows, is that given by Prandtl and Tietjens[31], and the author has improved upon this study at several points. (However, he has not included all the interesting material given therein as, for instance, the "length of transition" at the entrance, the intermittent occurrence of turbulence in the critical range, the effect of convergent and divergent flow, and the seventh-root law of velocity distribution.)

In studying through parts of the paper with a graduate class in fluid mechanics, one idea has occurred to the writer which is perhaps worth noting. In the paragraphs following Fig. 8 the author assumes a linear distribution of velocity in the boundary layer. This simplifies the mathematics, but is contrary to the fact shown in Equation (25) that for stream-line flow the velocity distribution is parabolic. As a matter of fact, the assumption is unnecessary. One can assume that the flow in the boundary layer obeys exactly the same laws as in the case of laminar flow, but that τ_0, the intensity of shear at the pipe wall, has the value of turbulent flow. Combining Equations (13) and (25):

$$v = \frac{f \rho V^2}{16 \mu} \left(\frac{r^2_0 - r^2}{r_0} \right) \qquad (81)$$

which will hold within the boundary layer only. Taking, as the author does, v_w as the velocity at the inner surface of the boundary layer, and δ as the thickness of the boundary layer, and substituting $v = v_w$, $r = \frac{D}{2} - \delta$, and $r_0 = \frac{D}{2}$ in Equation (81), and solving for δ:

$$\delta = \frac{D}{2} \left\{ 1 - \left(1 - \frac{32}{f \mathsf{R}} \frac{v_w}{V} \right)^{\frac{1}{2}} \right\} \qquad (82)$$

Expanding Equation (82) by the binomial theorem and dropping all except the first term gives Equation (36). The difference between the two expressions will be largest when $f \mathsf{R}$ is smallest; that is, just above the critical region, say, when $\mathsf{R} = 3\,000$. Then Equation (32) gives $f = 0.0428$ and $f \mathsf{R} = 128$. If $\frac{v_w}{V} = \frac{1}{2}$, as assumed by the author in his example, Equation (36) gives $\delta = 0.03125\, D$ and Equation (82) gives $\delta = 0.03229\, D$, or 3.3% greater thickness of boundary layer. For larger values of R the relative error is less, and the absolute error very much less. Thus, the author's assumption is satisfactory for purposes of calculation, but it is not necessary to think of the velocity in the boundary layer as varying uniformly.

One other fact may be added. If Equations (32) and (35) are solved simultaneously, $\mathsf{R} = 117\,500$; that is, Equation (32) may be taken as the law of fluid friction in smooth pipes up to a Reynolds' number of 117 500 and Equation (36) beyond that point. As both these equations are empirical, it is probable that there is actually no sudden change of law.

[31] "Applied Hydro- and Aeromechanics", Chapters III and IV, N. Y., McGraw-Hill Pub. Co., 1934.

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DISCUSSIONS

COMPARISON OF SLUICE-GATE DISCHARGE IN MODEL AND PROTOTYPE

Discussion

BY MESSRS. RAYMOND BOUCHER, AND H. E. HURST

RAYMOND BOUCHER,[4] JUN. AM. SOC. C. E. (by letter).[4a]—More than ever the hydraulic engineer is in need of proofs that model experiments are trustworthy. Mr. Blaisdell's paper is an excellent contribution toward that end.[4b]

The author states that "the coefficient, c_d, is not equal for all openings or heads." The statement leaves the reader under the impression that the head depends on the gate-opening, which is not the case in these experiments because by varying the discharge the gate-opening and the head were adjusted independently. It would be preferable to write that "the coefficient, c_d, varies with different gate-openings and different heads."

Referring to Table 1, the coefficient, c_d, corresponding to a gate-opening of 4.0 ft, has a difference of + 4.1% between model and Nature, whereas for gate-openings between 1.0 and 3.0 ft the difference in coefficient varies only from − 0.5 to + 1.0 per cent. The existence of that relatively abnormal difference in c_d, for the gate-opening of 4.0 ft, is not quite clear to the writer.

It is remarkable that the values obtained by model experiments are in very close agreement with those obtained in Nature. This paper is a valuable addition to the science of hydraulics and is another step in proving the reliability of model experiments.

H. E. HURST,[5] ESQ. (by letter).[5a]—In making a comparison of the discharge of a model of the Tremont sluice-gates with the discharge of its prototype, the author has done valuable work. It is to be hoped that the publication of his paper will lead to the recording of other comparisons of a like nature. He mentions the comparisons, made by Mr. Watt and the writer, of the discharges of models of the Assuan sluices with those of the actual sluices. In

NOTE.—The paper by Fred William Blaisdell, Jun. Am. Soc. C. E., was published in January, 1936, *Proceedings*. This discussion is printed in *Proceedings* in order that the views expressed may be brought before all members for further discussion of the paper.

[4] Asst. Prof. of Hydr., Ecole Polytechnique, Montreal, Que., Canada.

[4a] Received by the Secretary February 24, 1936.

[4b] That part of Mr. Blaisdell's paper entitled "Agreement with Froude's Law", is to be deleted from the paper when it is finally published with the discussion in *Transactions*, and, therefore, is not open to discussion.

[5] Director-General, Physical Dept., Ministry of Public Works, Cairo, Egypt.

[5a] Received by the Secretary March 11, 1936.

this connection it will be of interest to summarize some work that has been published[6] since the papers to which he refers.

The full-scale sluice discharge measurements at Assuan depend initially on volumetric measurements with a tank, and various methods have been adopted to extend these data to measure the discharge of sluices for which the tank cannot be used. As a result of this work, the discharge of the Nile can be measured by means of the sluices with a high degree of precision.

A comparison of these measurements with simultaneous measurements of the discharge by means of current meters has shown that during the low stage of the river the difference between the sluice and the current meter measurements is negligible. During the flood period the discharge given by the current meters was about 5% greater than the discharge given by the sluices.

The Assuan model experiments to which Mr. Blaisdell refers were made on sluices high above the river bed and discharging freely into air. For these experiments there was very good agreement between model and prototype. Since then experiments have been made on the type of sluice, 7 m (23.0 ft) high by 2 m (6.56 ft) wide, through which the river passes under a low head in flood time.

For these sluices the conditions are very different. They are not very high above the river bed and, therefore, are affected by differences on the river bed up stream of them. They are used when the velocity of approach is high and the direction of approach is different for different sluices. They are partly submerged down stream and the down-stream level is very difficult to measure in the turbulent condition of the water. It is also very different at different places owing to the large variation in the level of the bed down stream of the dam.

The mean discharge of six similar sluices is 6% less than the discharge inferred from the model. Two of these sluices are more similar in condition to the majority of the sluices than the other four, and for these two the mean discharge is only 1.7% less than that inferred from the model. The experiments on the model were very complete and covered a large range of up-stream and down-stream levels, and all measurements were made directly by means of a measuring tank.

A a result of these experiments three conditions of flow were found: (1) Free flow into the air; (2) submerged flow; and (3) intermediate types between Conditions (1) and (2).

(*1*).—Free flow into air persisted until the down-stream level was considerably above the sill of the sluice. In these conditions, the flow was represented by,

$$Q = c\,A\,\sqrt{2\,g\,(H - F)} \quad\text{.....................(2)}$$

in which Q is the discharge; A, the area of the gate-opening; H, the head above the sill of the sluice; and c and F are constants for any particular gate-opening.

[6] "The Measurement of the Discharge of the Nile through the Sluices of the Aswan Dam", by H. E. Hurst and D. A. F. Watt, *Physical Dept. Paper No. 24*, Ministry of Public Works, Egypt; also, "Further Experiments on the Discharge of Models of Sluices", by H. E. Hurst, *Physical Dept. Paper No. 25*, Ministry of Public Works, Egypt. (A copy of *Papers Nos. 24* and *25* are available for reference, at Engineering Societies Library, 33 West 39th St., New York, N. Y.)

(*2*).—The submerged condition begins when the down-stream level is well above the top of the sluice-opening, in which case the discharge is represented by,

$$Q = c\,A\,\sqrt{2\,g\,(H - h)} \quad \text{.....................(3)}$$

in which h is the head down stream above the sill.

(*3*).—The transition condition lies between the free and submerged conditions. The down-stream level affects the discharge, which, however, is not given by Equation (3). These conditions are best illustrated by reference to Fig. 5[7] in which a form of representation is adopted that is very useful in the study of the discharge of sluices or weirs.

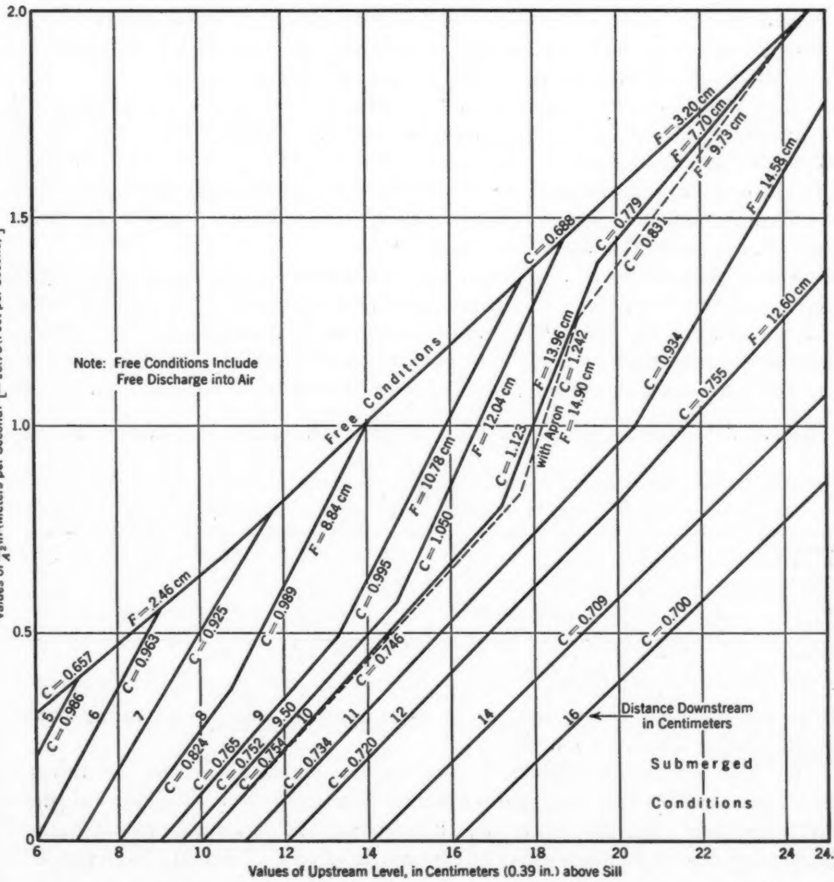

FIG. 5.—RELATION BETWEEN UP-STREAM LEVEL AND $\left(\frac{Q}{A}\right)^2$. (SLUICE, 4 CENTIMETERS, OR 1.57 INCHES, OPEN-MODEL SCALE, ASSUAN SLUICE, 1; 50; AND SLUICE, 14 BY 4 CENTIMETERS SQUARE).

[7] *Physical Dept. Paper No. 25*, Pl. 3.

In Equations (2) and (3), $\frac{Q^2}{A^2}$ is a linear function of H and, therefore, in Fig. 5, the two are plotted against each other. The result shows immediately the type of flow under any conditions of up-stream and down-stream level. The actual observation points are not shown, but they were numerous and fell very closely on the lines drawn in the diagram.

The conditions of flow at Assuan are usually those described as "free", and it is to these that the comparisons of model and prototype refer. It would seem advisable, if possible, to avoid the complicated transition conditions in using a model sluice as a means of inferring the discharge of its prototype, as it may be that the discontinuities do not occur always at the same points on both. For example, one of the Assuan type of sluices showed a condition in which there were two separate values of the discharge for the same head where only one could be obtained on the model, although this condition was actually in the region of free discharge into air.

The results of the Assuan model experiments led to the use of models as a means of inferring the discharge through the sluices of the dam on the Blue Nile, near Sennar. The results of this work are described in *Paper No. 25* previously mentioned, and from these results are constructed the tables of discharge for the Sennar Dam, which are the basis upon which the Sudan takes its share of the Blue Nile water.

Since these tables of discharge were constructed several years have elapsed during which discharges have been measured regularly with current meters down stream of the dam. The comparison of these discharges with the sluices during the important period when the Sennar Reservoir is working is given in Table 3. At this period the velocity of the water is not high, and conditions are favorable for current-meter measurements which were usually made about twelve times per month.

TABLE 3.—PERCENTAGE DIFFERENCES BETWEEN SLUICES AND CURRENT METERS, SENNAR DAM

Month	1929–30	1930–31	1931–32	1932–33	1933–34	1934–35	1935–36	Mean
November	−0.6	+1.0	+4.3	+4.0	+5.9	+10.4	+9.8	+5.0
December	+4.3	+0.2	+1.4	+2.1	+6.1	+9.8	+4.4	+4.0
January	+3.3	−3.5	−3.4	−2.3	+3.8	+3.6	+3.6	+0.7
February	−6.2	+2.6	−8.6	−7.3	−1.6	−2.6	....	−4.0
March	−5.9	+4.5	−5.3	−3.5	+1.0	−4.9	....	−2.4
April	+2.4	+9.0	−6.2	+1.2	+3.6	+8.2	....	+3.0
Means	−0.4	+2.3	−3.0	−1.0	+3.1	+4.1	....	+1.0

On the average the difference between the sluice discharges, as inferred from the models, and the current-meter discharges is negligible, but there are occasional months when the discrepancy is as much as 10%, and there seem also to be some signs of systematic effects. The data have not been investigated in detail; therefore, it is not possible at the moment to state to what extent the discrepancies are due to the sluices or to the current meters, but in any case the result is satisfactory as an example of the reliability of deductions from models. It should be mentioned that the conditions of flow were always those described as "free."

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DISCUSSIONS

BEHAVIOR OF STATIONARY WIRE ROPES IN TENSION AND BENDING

Discussion

BY MESSRS. C. D. MEALS, AND G. P. BOOMSLITER

C. D. MEALS,[5] ASSOC. M. AM. SOC. C. E. (by letter).[5a]—Another "link in the chain" has been wrought by Mr. Stewart in his splendid paper pertaining to the strength of stationary wire ropes looped over sheaves. Of the bending stress formulas, Equations (1) to (7) inclusive, it may be noted that Chapman's (5)[3] formula antedated Hardesty's (2)[3] formula; the former was published in 1908 and the latter in 1918; consequently, it should not be implied that Chapman modified Equation (3).

Equation (5) was developed by Josef Hrabak[6] and published in 1902. Hrabak's writings on the subject are frequently ignored, and yet he presented the first logical theory on the subject as compared to the Reuleaux formula in vogue in 1902. Howe's formula, Equation (6), was first published in 1907, although credit is generally given to his 1918 paper (2)[3]. The years 1902 to 1918 saw the publication of many formulas for the calculation of bending stresses in operating wire ropes, and there may have been some justification for the consideration given the subject, as ropes did break before being worn appreciably, which was considered as indicative of abnormal bending stresses.

With the present knowledge of designing wire ropes, it is appreciated that improper proportioning of the wires lead to their premature breaking. If less time had been spent on bending stress theories and more time devoted to the engineering design of the rope, the troubles experienced would have been greatly eliminated.

In discussing bending stresses in wire rope, the author of an article[7] published in 1930, noted that "the intensity of stress due to bending varies inversely as the radius of curvature; consideration of this fundamental fact

NOTE.—The paper by Douglas M. Stewart, Jun. Am. Soc. C. E., was published in February, 1936, *Proceedings*. This discussion is printed in *Proceedings*, in order that the views expressed may be brought before all members for further discussion of the paper.

[5] Wire Rope Engr., The B. Greening Wire Co., Ltd., Hamilton, Ont., Canada.

[5a] Received by the Secretary April 13, 1936.

[3] For reference to figures in parenthesis, see "Bibliography", Appendix II, of the paper, *Proceedings*, Am. Soc. C. E., February, 1936, p. 191.

[6] "Die Drahtseile", by Josef Hrabak.

[7] "Instructions for the Design of Wire Rope Installations", U. S. Navy Dept., Bureau of Construction and Repair, *Technical Bulletin No. 1*, p. 30.

leads to the conclusion that the wires in contact with the sheave or drum, *i e.*, those bent to the least radius of curvature, are subjected to the greatest bending stress." An extensive experience in testing moving wire ropes, under load, over one sheave and under another sheave, subjecting the rope to a reverse bending, has verified the foregoing statement.

In a discussion of Leffler's paper (3)[3], the writer noted that "the maximum bending stress [in an operating wire rope] is not necessarily in the outer wires of the strand farthermost from the axis of the rope." With some types of wire ropes and under certain conditions of loading and operation, however, the outer wires of the strand break next to the manila center of the rope where they are not susceptible to inspection.

For the determination of the strength of a wire rope bent over a sheave and subject to a static load, in a recent paper[8], the writer modified Equation (9) as follows,

$$S = k_1 A \epsilon \left\{ t - \frac{E_r D}{D + d_r} \right\} \ldots\ldots\ldots\ldots\ldots\ldots (14)$$

in which k_1 is a correction factor with the following values:

$\frac{D}{d_r}$	k_1	$\frac{D}{d_r}$	k_1
3	1.150	8	1.080
4	1.135	10	1.055
5	1.120	12	1.03
6	1.105	14	1.00
7	1.095		

and E_r is the modulus of elasticity of the rope as manufactured and as determined by the first run loading on it, the load not to exceed 30% of the strength of the rope.

The use of Equation (14) will increase the values given in Table 10, under the heading, "Equation (9)", and using the first-run modulus values given in Fig. 8. The changes in tabular values are given separately in

TABLE 11.—STRENGTHS OF ROPES BENT OVER SHEAVES

Set No	Equation (14) — Sheave Diameters, in Inches: 18	14	10	7
1	53 600	49 800	45 600	37 200
2	47 600	42 200	34 800	21 700
3	50 200	45 500	39 450	28 450
4	51 700	47 400	42 200	32 300
9	66 600	63 700	61 900	56 400
10	66 600	63 700	61 900	56 400
11	65 700	62 500	60 100	53 800
12	66 300	63 300	61 200	55 500
13	74 400	70 800	68 400	61 700

[8] "Main Cables and Suspenders for Suspension Bridges", by C. D. Meals, Assoc. M. Am. Soc. C. E., *Journal*, Eng. Inst. of Canada, August, 1934.

Table 11. These values show an appreciable advance compared with the data given in Table 10 for Equation (9); if the loadings on the ropes had not been so abnormally high, lower first-run moduli would have resulted, with a corresponding increase in the values of Table 11.

It should be appreciated that Equation (14) is only an approximation, and yet it gives values quite closely in accord with the results of Skillman's (9)[3] and Rairden's (1)[3] series of tests and also with the test results of many 6 × 19 and 6 × 37, steel-center, suspender ropes as used on recent suspension bridges; although, as indicated previously, it is not in as close agreement with Mr. Stewart's test results.

For 6 × 19 ropes with manila centers, the efficiencies of the ropes reported by Mr. Stewart are higher than those of Skillman's (9)[3] tests for 6 × 19 Warrington plow-steel ropes and of Rairden's (1)[3] tests of 6 × 19 filler-wire improved plow-steel ropes, and it appears from a comparison of these three series of tests that the efficiencies may vary for the different types and grades of 6 × 19 ropes and even for ropes of the same type as made by the different manufacturers; consequently, too much reliance must not be placed on any particular formula until more tests are conducted to verify their accuracy, although no brief is being held for Equation (14) as the writer appreciates its limitations.

Equation (10) has been used for a number of years to determine the strengths of special wire ropes and has found to be more accurate than is indicated in Table 9. To verify this statement, tests were made of $\frac{7}{8}$-in. and 1-in. ropes with manila centers, the data pertaining to the ropes being noted in Table 12, and the results of these tests in Table 13.

TABLE 12.—DESCRIPTION OF $\frac{7}{8}$-INCH AND 1-INCH WIRE ROPES

Set No.	Description of ropes	Metallic area, in square inches	ANGLES		
			a	a_s	b
14.....	$\frac{7}{8}$-in., 6 × 7 Lang lay plow-steel, non-preformed	0.31975	14° 15′	12° 13′	15° 13′
15.....	1-in., 6 × 9 filler-wire regular lay cast-steel, non-preformed..............................	0.41268	17° 28′	14° 4′	18° 50′
16.....	1-in., 6 × 19 filler-wire regular lay plow-steel, non-preformed..........................	0.41268	17° 28′	14° 4′	18° 50′
17.....	1-in., 6 × 19 filler-wire regular lay plow-steel, preformed............................	0.41268	16° 44′	13° 28′	19° 10′

The ropes were tested by the Ontario Department of Mines, in Toronto, Ont., Canada, and the individual wires of the ropes were check-tested by the Steel Company of Canada, Limited, at Hamilton, Ont. It will be seen from Table 13 that Equation (10) does agree quite closely with test values, and it is difficult to reconcile these efficiencies with those noted by Mr. Stewart in Table 9.

One criticism of Equation (12) is that it does not take into consideration the difference in the angles of lay of the various wires in a strand. For example, it will give the same efficiency for a 6 × 19 two-operation strand, a strand having 12 wires laid over 7, as for a 6 × 19 one-operation strand rope

as a filler-wire construction; and yet it must be obvious that the efficiency of the latter is greater than that of the former construction. Equation (10) makes this differentiation, whereas Equation (12) does not; also, the latter part of Equation (10) will give the breaking strength of the individual strands of the rope quite accurately.

TABLE 13.—ACTUAL AND CALCULATED BREAKING STRENGTHS AND EFFICIENCIES OF WIRE ROPES

Set No.	Actual Tests		Equation (10)		Equation (12)		Equation (16)	
	Load, in pounds	Efficiency (percentage)	Load, in pounds	Efficiency (percentage)	Load, in pounds	Efficiency (percentage)	Load, in pounds	Efficiency (percentage)
14.....	66 050	90.2	65 300	89.1	63 750	87.1	65 000	88.8
15.....	67 285	85.6	67 500	86.0	63 300	80.6	66 000	84.0
16.....	83 425	85.4	83 700	85.7	78 800	80.6	82 000	84.0
17.....	81 225	85.9	81 900	86.6	76 600	81.0	79 600	84.2

Equation (12) may be modified to take into consideration the varying lays of wires in the strand by taking a as the average angle of lay of all the wires in the strand, or $a_a = \sum \frac{na}{n}$, which, for a 19-filler wire strand, becomes,

$$a_a = \frac{6\,a_1 + 6\,a_2 + 12\,a_3}{25} \qquad (15)$$

and, accordingly, Equation (12) may be written:

$$S = A\,S_w \cos\,(a_a + b) \qquad (16)$$

Values in accordance with Equation (16) are noted in Table 13.

It is regrettable that certain pitfalls were not avoided in Mr. Stewart's tests inasmuch as they detract somewhat from the value of the paper. Among others, four may be noted, as follows: (*a*) Loadings at 49% of the rope strength for the determination of the modulus of elasticity values; (*b*) a decidedly short gauge length of 10 in. for the measurement of the stretch of the rope under load; (*c*) the use of fixed clamps on the rope; and (*d*) the adoption of ropes with manila centers.

(*a*).—*Loadings at 49% of the Rope Strength for the Determination of the Modulus of Elasticity Values.*—Most certainly this is an abnormally high loading and not representative of any engineering or commercial practice pertaining to wire rope; lower loadings are more in keeping with actual practice and would result in lower modulus values. Such abnormally high loads result in a permanent breaking down of the structure of the manila center, nicking of strand against strand, and a high modulus value that is unreal in so far as standard practice is concerned.

(*b*).—*A Decidedly Short Gauge Length of 10 Inches for the Measurement of the Stretch of the Rope Under Load.*—It has been general practice in modulus tests of wire ropes, to use a gauge length as long as possible; Skillman (9)[8] used a gauge length of 50 in., and for most of the suspender ropes as

used on the large suspension bridges built in recent years, the gauge length has been 80 in. The merit of the longer gauge length is that any irregularities or errors in measurements or in the behavior of the rope are not proportionately of much consequence as they must be in the shorter gauge lengths; those experienced in such testing will appreciate this point.

(*c*).—*The Use of Fixed Clamps on the Rope.*—Special swivel clamps have been used that are free to swivel or rotate, and, consequently, to eliminate any twisting of the measuring apparatus due to the untwisting of the rope.

(*d*).—*The Adoption of Ropes with Manila Centers.*—That such ropes are used to avoid confusion in the mathematical analyses is appreciated, but they are not as typical as those with an independent wire-rope center (IWRC); particularly for the consideration of the loss in strength due to bending, as this applies to suspender ropes for suspension bridges and such ropes are always made with an independent wire rope center.

It is a fact fairly well known to wire-rope engineers, that the tensile strengths of preformed wire ropes are from 3 to 5% lower than the strengths of non-preformed ropes, but sales policies have conveniently "glossed over" this fact. For the preforming of Lang lay wire ropes, the use of quills as described by Mr. Stewart is not necessary, as the roller head shown in Fig. 4 may be used; in fact, roller heads only are used in the making of all types and diameters of preformed Lang lay wire ropes by the writer's Company.

Relative to the coefficients of friction given in Table 6, it is presumed that these are for ropes that were dry—that is, devoid of any heavy lubricant. Mr. William Hewitt published data[9] pertaining to this subject in 1905, and it may be interesting to compare his values with those of the author.

That the bending stress in a Lang lay rope is approximately 20% less than that in a regular lay rope verifies a statement that the writer[10] made in 1928 regarding such ropes.

Mr. Stewart shows the same confusion as did Carstarphen and Rairden in the paper cited (1)[8] in considering that the loss of strength of a stationary wire rope looped over a sheave is the same as the bending stress in a moving wire rope operating over a sheave; the former is more susceptible of mathematical analyses than the latter. Reasons and examples were cited by the writer in his discussion of Carstarphen's paper (1)[8] to indicate that the latter was not susceptible to such an analysis and surely not to the extent that a "prediction of the bending stress" could be satisfactorily assured, as noted in Conclusion (10) of the paper. It would be a boon to the wire-rope users as well as to the manufacturers if such a prediction was possible.

G. P. Boomsliter,[11] M. Am. Soc. C. E. (by letter).[11a]—In calling attention to the increase in the modulus of elasticity of a wire rope under successive applications of load, Mr. Stewart has rendered a valuable service. A value of E of 18 000 000 lb per sq in. is not too great. As the author has shown, this value is perhaps high for repetitions of load under the proportional limit

[9] "Elements of Machine Design", by O. A. Leutwiler.

[10] "Aerial Tramways", by F. S. Carstarphen, M. Am. Soc. C. E., *Transactions*, Am. Soc. C. E., Vol. 92 (1928), p. 964.

[11] Prof. of Mechanics, West Virginia Univ., Morgantown, W. Va.

[11a] Received by the Secretary April 16, 1936.

but, at some time or other during their period of use, most hoisting ropes are stressed beyond the calculated load. The writer is reminded of two such cases in his brief experience with wire ropes.

In one case a mine cage was customarily left all night at the bottom of a 250-ft shaft down which came the fresh air draft to a mine. One cold night the cage froze fast to the floor. It was finally pulled loose by stressing the hoisting rope, but the rope was 8 ft longer after pulling it loose than it was before. In another case, the circuit breaker on a hoist went out as a load of coal was being lifted in a shaft. A telemeter attached to the hoisting rope immediately above the cage showed that the action of the safety devices in stopping the cage caused stresses which were 2.29 times the dead load stresses. Many other conditions result in occasional applications of high stress to a rope so that after a short period of service its modulus of elasticity has been increased beyond that of a new rope. Indeed, calculations made in the telemeter test referred to, indicated a modulus of elasticity for the rope there tested of between 19 000 000 and 20 000 000 lb per sq in.

Mr. Stewart deserves congratulations for his clever method of attaching the tensometers to his wire rope when it was bent about a sheave. He has pointed the way for further investigations of bending stress in wire rope. However, the results of bending tests such as those of this paper are likely to be misleading. Undoubtedly, they determine the stresses due to bending a wire rope about a thimble, or in a stationary rope bent over a sheave while in an unstressed condition, since there is no constraint as the wires adjust themselves to the curved position about the sheave, but a heavily loaded rope running over a sheave will have other stresses which the author has not considered. These stresses are due to the frictional resistance to sliding of the wires upon each other when the loaded straight rope bends about the sheave. To illustrate, consider an axial load of 30 000 lb on one of Mr. Stewart's 1-in. ropes of regular lay as it passes over a sheave. Each strand of 19 wires will be assumed to take one-sixth of the load, or 5 000 lb.

The lay length of the strand will be taken as 93 d and d as $\frac{d_r}{15}$, in which d_r = the diameter of the rope; and d the diameter of the individual wires. The lay length will then be $\frac{93}{15}$, or 6.2 in. and the length of half a lay will be 3.1 in. The lay angle is 18° 39′. The component of stress in a strand normal to the axis is 5 000 tan 18° 39′ = 1 688 lb.

Let Fig. 18(b) represent a half lay length of the strand. Fig. 18(a) shows the components of the stresses at the two ends of the length which are normal to the strand length. These components are held in equilibrium by pressures of the other strands, assumed normal to the strand in question. By analogy with the pressures and tensions in a cylindrical vessel, the total lateral pressure in a length of a half lay will be 2 × 1 688 = 3 375 lb. Now, assume that the lower end of this length is in contact with the sheave. The upper end will be at the outside of the rope. The part in contact with the sheave will shorten and the part at the outside of the rope will lengthen. This is

done by the slipping of the strand on its neighbors, but this slipping will be done against a friction. Assuming a coefficient of friction of 0.15, the frictional force set up to oppose this motion in a half lay length will be 506 lb. This force will be a measure of the difference between the stress in this strand at contact with the sheave and at the outside of the rope. Note that this is 10.3% of the axial stress in the strand.

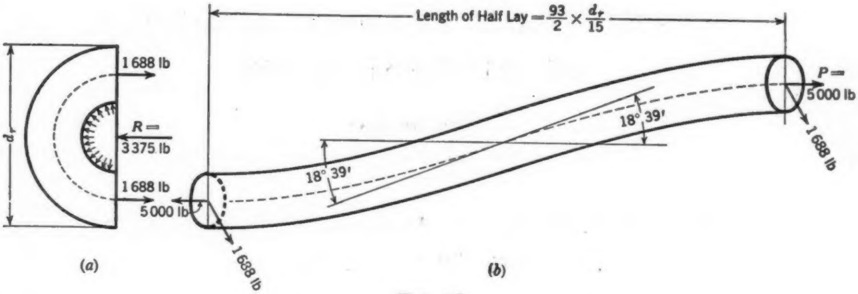

FIG. 18.

If the normal length of the rope is maintained at its line of contact with the sheave, all this frictional restraint (506 lb), will measure the increase in strand tension at the outside of the rope. If there is slip on the sheave the normal length of the rope is maintained along a line somewhere between the sheave and the outside of the rope. Assuming that this line coincides with the rope center, the stress at the inside is decreased and that at the outside increased, each by one-half the frictional restraint, or 253 lb. The area of the wire in one strand, as taken from Table 3 of the paper, would be 0.0694 sq in. The unit stress due to direct load would be 72 000 lb per sq in., and the frictional bending stress according to the first assumption would be 7 490 lb per sq in., and 3 745 lb per sq in., according to the second. These stresses, of course, are in addition to the stresses due to flexural bending. They would be independent of the ratio of the sheave diameter to the rope diameter. The formula expressing this stress would be:

$$s_f = 2\, s_R \tan \alpha\, f \qquad (17)$$

in which s_R is the axial unit stress in the rope; α is the angle of lay; and f is the coefficient of friction between strands. Since the same condition exists between the wires in a strand, Equation (17) is decidedly approximate and is given simply to indicate the effect of frictional resistance on sliding. The coefficient of friction is also assumed. Only further tests will indicate what it actually is, but the writer is firmly of the opinion that tests such as those presented by Mr. Stewart are likely to be misleading if assumed for a heavily stressed rope passing over a sheave. If a rope were rusted so that it could not slip, it would bend as a unit, of course, and Equation (7) would be a proper formula for bending stress. If it were so well lubricated that the friction was that of an oil layer on oil, this stress could be neglected. It is very doubtful whether this last condition would exist in a rope under heavy service. It is more likely that neglect would make f more than 0.15.

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DISCUSSIONS

VARIED FLOW IN OPEN CHANNELS OF ADVERSE SLOPE

Discussion

BY MESSRS. H. E. VON BERGEN, W. E. HOWLAND, AND ARNO T. LENZ

H. E. VON BERGEN,[16] JUN. AM. SOC. C. E. (by letter).[16a]—The science of obtaining the depth directly at any point in a channel of adverse grade and uniform section, for steady non-uniform flow, has been advanced appreciably by this paper. In the field of hydraulics, however, this case is rather rare and in the event of flow below critical depth the hydraulic jump is produced. Furthermore, unless the engineer has occasion to make numerous computations, using the author's formulas and "varied flow functions", he may find the method somewhat involved and cumbersome.

The writer has found it more practical and convenient to dispense with the author's analytical treatment beyond his Equation (3) and, solving for $\frac{dy}{dx}$, to make a graphical integration of the resulting differential equation as suggested[17] by Harold A. Thomas, M. Am. Soc. C. E. For example, using the notation of the paper:

$$\frac{dy}{dx} = \frac{-S_0 - \dfrac{Q^2}{K^2(y)}}{1 - \dfrac{Q^2\, b}{g\, a^3}} \qquad (16)$$

and, since $\frac{Q^2}{K^2}(y) = S$, which may be obtained quickly from various prepared

NOTE.—The paper by Arthur E. Matzke, Jun. Am. Soc. C. E., was published in February, 1936, *Proceedings*. This discussion is printed in *Proceedings*, in order that the views expressed may be brought before all members for further discussion of the paper.

[16] With U. S. Forest Service, Sacramento, Calif.

[16a] Received by the Secretary March 17, 1936.

[17] "Hydraulics of Flood Movements in Rivers", by Harold A. Thomas, *Engineering Bulletin*, Carnegie Inst. of Technology, 1934, p. 13.

charts or tables, and $\frac{Q^2 b}{g a^3} = \frac{V^2 b}{g a}$ (and for rectangular channels may be further simplified) Equation (16) may be reduced to:

$$\frac{dy}{dx} = \frac{-S_0 - S}{1 - \frac{V^2 b}{g a}} \qquad (17)$$

To integrate, graphically, use the reciprocal of Equation (17), compute four or five values of $\frac{dx}{dy}$ over the desired range of depths, and plot the depth, y, as ordinate and $\frac{dx}{dy}$ as abscissa. Then, by graphical summation of the area to the left of the curve, the water-surface profile may be drawn over the desired range of depths.

The negative sign of S_0 was introduced from Equation (3), but for the general case the sign of S_0 is positive and, by substituting a minus value for "adverse" slope and a positive value for "sustaining" slope, the sign of S_0 will take care of itself automatically.

The writer believes the graphical solution is more practical and efficacious and, except for accidental errors in plotting, is more correct than the elaborate analytical treatment given by the author.

W. E. Howland,[18] Assoc. M. Am. Soc. C. E. (by letter).[18a]—This extension of the theory of varied flow to open channels of "adverse" slope furnishes additional proof of the power of the fundamental methods presented by Professor Bakhmeteff[2], and is, in itself, a valuable contribution to the practical tools of the hydraulic engineer. The concise presentation of theory and the numerous illustrative problems serve to make the methods available to students and practitioners alike.

A few suggestions may be helpful in gaining familiarity with the terminology. When the depth is y, the term, $K(y)$, is defined as $C\,A\,\sqrt{R}$, in which C is the constant in the familiar Chezy formula, $V = C\sqrt{R\,S}$; R is the hydraulic radius; and A is the area of the cross-section of the moving stream. It is thus a part of the total expression for the carrying capacity of the channel, but not all of it. The term, "conveyance, as a name for $K(y)$ is a good one, because it is not used for any other quantity, but the expression, "carrying capacity", seems to mean Q, or an expression for Q, and, therefore, may be misleading as applied to $K(y)$.

The term, "hydraulic exponent", or n, can be defined by the equation $K^2(y)$ = a constant $\times\ y^n$. It is obtained by plotting the logarithms of the computed values of $K^2(y)$, for a given cross-section against y, the depth, as abscissa. The slope of the line so plotted is the value of n.

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[18] Asst. Prof. of San. Eng., Purdue Univ., La Fayette, Ind.

[18a] Received by the Secretary March 27, 1936.

[2] "Hydraulics of Open Channels", Eng. Societies Monograph, McGraw-Hill Co., 1932.

There are two assumptions made in Mr. Matzke's study which might introduce small errors. A useful addition to the paper would be an estimation of the approximate limits of these errors as affecting computed depths in one or more of the illustrative problems. These two assumptions are: (1) That of the constancy of n previously mentioned; and (2), that of the constancy of η defined in the paragraph following Equation (4).

It is interesting to note that a more elementary method of solving problems involving changing velocities does sometimes yield results of sufficient precision with little labor. The writer has used one such method to check the results of Example 1. It involves the consideration of the canal as made up of six sections of variable length, but of constant difference in surface elevation of 1 ft. The effect on the surface curve due to the differences in velocity head can be considered exactly. The effect on the surface curve of friction in each section was obtained by computing the value of the energy gradient at the extremity of each section and then using the harmonic average (a reciprocal average of reciprocals) of the two extreme values in estimating the effect of friction in that section. The variable length of each section is then computed from the following self-evident expression (the nomenclature used is that of the paper):

$$l_{1\text{-}2} = \frac{\text{Difference in depths (1-2)} - \text{difference in velocity heads (2-1)}}{\text{Bottom slope} + \text{energy gradient}}$$

or,

$$l_{1\text{-}2} = \frac{(y_1 - y_2) - \frac{Q^2}{2\,g}\left\{\frac{1}{(a_2)^2} - \frac{1}{(a_1)^2}\right\}}{s + \frac{Q^2}{k^2_{(\text{ave.})}}} \quad \ldots\ldots\ldots\ldots(18)$$

For Example 1, Equation (18) becomes:

$$l_{1\text{-}2} = \frac{1 - \frac{519^2}{64.4}\left\{\frac{1}{(a_2)^2} - \frac{1}{(a_1)^2}\right\}}{0.0004 + \frac{519^2}{k^2_{(\text{ave.})}}} \quad \ldots\ldots\ldots\ldots(19)$$

The values of a and k have been obtained for this section from Professor Bakhmeteff's work[13]. Substituting the values of $a_2 = 58.5$ ft^2 when $y = 3$ ft. and $a_1 = 84.0$ ft^2 when $y = 4$ ft, the numerator of Equation (19) becomes 0.373. Using the average of the values of $k^2 = (58.2 \times 10^2)^2$, which is the value when $y = 3$ ft, and of $k^2 = (99.4 \times 10^2)^2$, which is the corresponding value when $y = 4$ ft, the denominator of Equation (19) becomes 0.004062 and $l_{1\text{-}2}$ is 84 ft. Owing to the very small differences in the functions given in Table 2, it is almost impossible to check this value thereby.

Likewise, the writer has computed the distance from the place where $y = 4$ ft to where $y = 5$ ft, etc., and then, by addition, the distance from where $y = 3$ ft to where $y = 5$ ft, etc., and obtained the results listed in

[13] "Hydraulics of Open Channels", Eng. Societies Monographs, 1932, Canal Type D, p. 321.

Table 3. The value, 426 ft, does not agree with the one given by Mr. Matzke, but it is more closely in agreement with the arithmetic values he himself has given; that is, $\frac{6.56}{0.0004}\left\{-(0.457-0.762)+1.06\,(0.452-0.715)\right\}=426.$ The agreement between the results of the two methods is striking and really surprising, considering that the approximate method involves so little work.

TABLE 3.—COMPARISON OF HORIZONTAL DISTANCES (UNITS IN FEET)

Depth at Sections Between Which Horizontal Distances Are Given:		Computed Horizontal Distances	
From:	To:	By approximate method:	By Matzke's method:
3	4	84	.....
3	5	425	426
3	6	1 170	1 180
3	7	2 320	2 335
3	8	3 830	3 860

It is also interesting to note that if the arithmetic average of energy gradients (or what amounts to the same thing, the average of $\frac{1}{k^2}$) were used in the foregoing computation instead of the harmonic average of energy gradients (or average value of k^2) at the two extremities of a given section, the agreement would not have been nearly so good. This would suggest that, at least for this type of problem, the harmonic average should be used.

From the application of this particular approximate method to this problem and also from studying the somewhat similar graphical method presented by H. Addison[19], the writer is led to believe that the harmonic average is preferable to the arithmetic average of slopes in approximate methods of this nature. If further study on this point were thought to be desirable, Mr. Matzke's method would serve as an exact standard of comparison on regular channels. In natural channels with irregular sections, in which some kind of an approximate method must be used, great precision could be secured by the use of a large number of sections or small increments of depth.

The writer has attempted to apply this approximate method to Example 2, but has obtained a result of 127 ft instead of 160 ft. Mr. Matzke's result might be in error almost to this extent. A change in the last significant figure of one of the numbers from Table 2 would change the result of his computation as much as 13 per cent. It seems that this table is not satisfactory when applied to sections so close to one another. Indeed, this lack of precision in computations with close sections is the most serious limitation to the application of the method. The approximate method increases in precision as the sections become closer and would appear to be more satisfactory for these cases.

Applying the approximate method to the third problem the writer has obtained a value of $l = 11\,150$ ft from the right-hand end of the canal to

[19] "Hydraulics", by H. Addison, N. Y., John Wiley & Sons, 1934, p. 135.

the place where $y = 10$ ft. (In making these computations a value of $k^2 = 19.35 \times 10^8$, when $y = 9$ ft., and $k^2 = 23.95 \times 10^8$, when $y = 10$ ft., was used, in addition to the values for the lower depths given by Bakhmeteff. Then, subtracting from 10 ft the value, 11 150 − 10 000, times the value of $\frac{dy}{dz}$ at the 10-ft depth, or $1\,150 \times 0.000425 = 0.49$, the writer obtains 9.51 ft as the depth of the water at the lower end of the canal. (Here, the water-surface slopes are very small and are very nearly constant.) This agrees very well with the values of 9.56 ft given by the author. This method probably involves no more labor than that of Mr. Matzke in solving this problem.

In the statement of this problem it is to be noted that the upper end of the canal level is held fixed at 3 ft; and that the reservoir surface elevation at Section *B* is invariable. It is impossible to maintain both conditions when the water is flowing to the left since the water line in the canal must be at least one velocity head lower than that in the reservoir. This velocity head at the highest flow amounts to 0.61 ft. The solution given is consistent with the assumption that the levels indicated at Sections *A* and *B* are canal levels and not reservoir or ocean levels, unless, in some case, they might happen to be the same.

For publication in *Transactions*, Equation (12) of this paper will be corrected to read as follows:

$$\int \frac{d\,\eta}{\eta^4 + 1} = \frac{1}{4\sqrt{2}} \log_e \left(\frac{\eta^2 + \eta\sqrt{2} + 1}{\eta^2 - \eta\sqrt{2} + 1} \right) + \frac{1}{2\sqrt{2}} \Big[\text{arc tan } (\eta\sqrt{2} + 1) + \text{arc tan } (\eta\sqrt{2} - 1) \Big] \quad \ldots\ldots (12)$$

ARNO T. LENZ,[20] JUN. AM. SOC. C. E. (by letter).[20a]—An interesting extension of the work of B. A. Bakhmeteff, M. Am. Soc. C. E., on open channels is contained in this paper. The examples show its practical application in an excellent manner. In mathematical derivations of this type, however, there are certain basic assumptions which must be made. These assumptions may be said to be the foundation upon which the structure is built, and they must be sound. They should be checked in every manner possible.

In the derivation of the general differential equation for varied flow, the hydraulic exponent, n, is used, based on the fact that "the conveyance function, $K = a\,C\sqrt{R}$, within a reasonable range of depths, follows sufficiently close the exponential relations: $K^2(y) = a^2 C^2 R = \text{constant } y^n$"[8]. This exponential relation is of extreme importance in all that follows.

It is evident that the best check of this relation would be to plot the conveyance, K, against the depth, y, on logarithmic paper, which plotting should give a straight line. Unfortunately, practically all published experimental

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[20] Instr., Dept. of Hydr. and San. Eng., Univ. of Wisconsin, Madison, Wis.

[20a] Received by the Secretary April 18, 1936.

[8] "Hydraulics of Open Channels", by B. A. Bakhmeteff, M. Am. Soc. C. E., Eng. Societies Monograph, 1932, p. 84.

discharge measurements omit the slope measurement necessary to determine the value of K from the discharge, $Q = K s^{\frac{1}{2}}$.

An approximate check might be made, however, by considering that when a number of stream gagings are taken, some will be made at times when the surface slope is greater than normal and at others when it is less than normal, with the average of a number probably about normal. Furthermore, the gage height as recorded is a measure of the depth, y. With these approximations in mind, discharge was plotted against gage height for ten streams picked at random from the *Water Supply Papers* of the U. S. Geological Survey, from all sections of the country. The curves are plotted on Fig. 6.

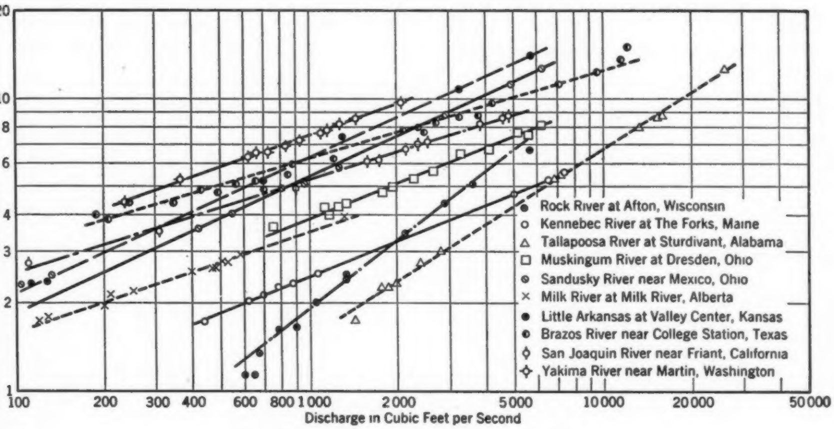

FIG. 6.—DISCHARGE RATING CURVES.

The writer was surprised that the points plotted did approximate straight lines. It is recognized that for canals it is likely that the exponential law holds even more closely than for natural channels. Therefore, the curves are a logical proof of the validity of the use of the hydraulic exponent as assumed by the author.

When flow spreads out over flood-plains there will be a sharp break in the line, and care must be exercised under such conditions. A check such as that presented herein is relatively easy to make and will show quickly whether the theory may be applied to any particular river or canal.

The author is to be congratulated on his extension of the theory presented by Professor Bakhmeteff, which should be a tool of great usefulness to those working on the hydraulics of open channels.